COMPLEX MANIFOLDS

Lent term

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prerequisites: differential geometry
basic complex analysis (holomorphic functions)

Remarks: Riemann surfaces are useful but not essential
Algebraic geometry is related, but not really used.

Books: Huybrechts "Complex geometry"
Griffiths & Harris "Principles of algebraic geometry" (CH0 and CH1)

There will be printed notes to come later, but are not a replacement for lecture notes.

4 example sheets & classes - first to come out on 22nd of Jan.

1. Introduction

Recall: smooth, real, n-dimensional manifold M - Hausdorff and second-countable, covered by charts: homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$, where $U_\alpha \subset M$ is open and connected, $V_\alpha \subset \mathbb{R}^n$ with $M = \bigcup_{\alpha \in A} U_\alpha$, $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth (C^∞) on its domain $\subset \mathbb{R}^n$.

Basic idea: replace $\mathbb{R}^n$ with $\mathbb{C}^n$, and C^∞ with holomorphic (also known as complex analytic), to obtain a "holomorphic structure".

We will need some basics of several complex variables.

Recall first about one complex variable. Let $U \subset \mathbb{C}$ be open, suppose $f : U \to \mathbb{C}$ is smooth in the $\mathbb{R}$-sense. We say f is **holomorphic** iff

- it is complex analytic (represented by some convergent power series, $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ valid for $\{|z-a| < \varepsilon\} \subset U$)
- it is complex differentiable ($\mathbb{R}$-smooth + Cauchy-Riemann equations hold:
  - ↳ if $z = x+iy$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$

Cauchy-Riemann: $\frac{\partial f}{\partial \bar{z}} = 0$ on U

A general smooth $f(z) = f(0) + \frac{\partial f}{\partial z}(0) z + \frac{\partial f}{\partial \bar{z}}(0) \bar{z} + o(\bar{z})$ $(a=0)$

the differential $df = \partial f + \bar{\partial} f := \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$ as $z \to 0$ $(dz = dx + idy,\ d\bar{z} = dx - idy)$.

Cauchy Integral formula: $f(z) = \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f(w)}{w-z} dw$ $\{|w-z|<r\} \subset U$

Now let $U \subset \mathbb{C}^n$ be open, and $f : U \to \mathbb{C}$ be C^1-differentiable in the real sense. Then **f is called holomorphic** if $g_j(z) := f(z_1, \dots, z_{j-1}, z, z_{j+1}, \dots, z_n)$ (fix all but j) is holomorphic in z $\forall j = 1, \dots, n$.

i.e. $\frac{\partial f}{\partial \bar{z}_j} = 0$ on U $j = 1, \dots, n$, where $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}\right)$ $z_j = x_j + iy_j$.

A shorthand for this will be $\bar{\partial} f = 0$.

N.B. it is often convenient to set U as a polydisc: $\Delta_1 \times \dots \times \overset{\text{standard disc} \subset \mathbb{C}}{\Delta_n} = \{ z \in \mathbb{C}^n : |z_j - a_j| < r_j \ \forall j\}$

$a_j \in \mathbb{C}$, $r_j > 0$

Cauchy Integral Formula: if $f : \underbrace{\overline{\Delta_1 \times \dots \Delta_n}}_{\Delta} \to \mathbb{C}$ is holomorphic, then

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\substack{|w_j - a_j| = r_j \\ \forall j}} \frac{f(w)}{(w_1 - z_1) \dots (w_n - z_n)} dw_1 \dots dw_n \qquad \forall z \in \Delta \qquad w = (w_1, \dots, w_n)$$

N.B. We're integrating over submanifold $\subsetneq \partial \Delta$ when $n > 1$.

Proof (gist) can do repeated integration in each w_j, $j = 1, \dots, n$, and treat $w_{j+1}, \dots, w_n$ as parameters.

Power series: holomorphicity $\Leftrightarrow$ $f(z) = \sum_{i_1,\dots,i_n=0}^{\infty} \frac{\partial^{i_1+\dots+i_n} f}{\partial z^{i_1} \dots \partial z^{i_n}} (a) (z_i - a_i)^{i_1} \dots (z_n - a_n)^{i_n} \frac{1}{i_1! \dots i_n!}$

If $f = (f_1, \dots, f_m) : \underset{\text{open}}{U} \subseteq \mathbb{C}^n \to \mathbb{C}^m$, then f is holomorphic iff each f_j is too. It is called <u>biholomorphic</u> if it is bijective and f and f^{-1} are both holomorphic.

Complex Jacobian : of a holomorphic function $f = (f_1, \dots, f_m)$, then

$$J(f)_z := \left(\frac{\partial f_k}{\partial z_j} (z) \right)_{\substack{k=1,\dots,m \\ j=1,\dots,n}}$$

defines a $\mathbb{C}$-linear map $\mathbb{C}^n \to \mathbb{C}^m$. If $J(f)_z$ is surjective, then we say z is a <u>regular point</u> of f. If $w \in \mathbb{C}^m$, then if $\forall z \in f^{-1}(w)$ is regular, then we call w a <u>regular value</u>.

Suppose $m = n = 1$, and write $f = u + iv$, f is holo. Then $J_{\mathbb{R}}(f) = \underset{\substack{\text{thought of as} \\ \text{a complex matrix}}}{\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}} \underset{\text{similar}}{\sim} \begin{pmatrix} \frac{\partial f}{\partial z} & 0 \\ 0 & \frac{\partial \bar{f}}{\partial \bar{z}} \end{pmatrix}$

i.e. $\forall \alpha, \beta \in \mathbb{R}$ $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \sim \begin{pmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{pmatrix}$

We can extend this to dimensions $m, n \geq 1$.

For a general function, $J_{\mathbb{R}}(f)$ regarded as a complex matrix is similar to $\begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix}$ for any holo. f.

When $m = n$ are equal, we get a square matrix, and

$$\begin{aligned} \det(J_{\mathbb{R}}(f)) &= \det(J(f)) \det(\overline{J(f)}) \\ &= |\det(J(f))|^2 \geq 0 \end{aligned}$$

In particular, > 0 when $J(f)$ is non-singular.

(Holo) Inverse function Theorem: if $U, V \subseteq \mathbb{C}^n$ open and $f : U \to V$ holomorphic with $z \in U$ a regular point, then $\exists$ nbhood U_0 of z s.t f maps U_0 biholomorphically onto it's image.

Dfn: a **complex n-fold** M is a hausdorff second countable topological space with complex coord. charts: homeomorphisms $\varphi_i : U_i \subseteq M \to \varphi_i(U_i) \subseteq \mathbb{C}^n$, where both U_i and its image are open and connected sets, such that $\forall i, j$, $\varphi_i \circ \varphi_j^{-1}$ are holomorphic on $\varphi_i(U_i \cap U_j)$, and $M = \bigcup_i U_i$.

If $p \in U_i$, $\varphi_i(p) = (z_1, \dots, z_n)$ are **complex local coordinates.**

Remark : we can think of M as a real $2n$-dimensional manifold with a choice of "holomorphic atlas": $\{(\varphi_i, U_i)\}$. Often referred to as the underlying real smooth manifold.

Dfn: let M, N be two complex manifolds with complex atlases $\{(\varphi_i, U_i)\}$ and $\{(\psi_\alpha, V_\alpha)\}$. If $F : M \to N$ is a continuous map, we say F is **holomorphic** if $\forall i, \alpha$ $\psi_\alpha \circ F \circ \varphi_i^{-1}$ is holomorphic as a complex function on it's domain $\varphi_i(U_i \cap F^{-1}(V_\alpha)) \subseteq \mathbb{C}^n$ check this is domain.

Manifolds M and N are **biholomorphic** (isomorphic) if $\exists$ a biholomorphic map $F : M \to N$.

(in fact, it suffices that if F is a holomorphic bijection, then F^{-1} is automatically holomorphic)

see complex variable notes for dim = 1 case

Proposition : Let M be a compact complex manifold. Then a holomorphic function $f : M \to \mathbb{C}$ is constant.

proof: Consider $|f| : M \to \mathbb{R}$. This is continuous since f is holo. Since M is compact, $\Rightarrow$ M attains a maximum, say at $p \in M$. Consider a chart $\varphi : U \longrightarrow \Delta \subset \mathbb{C}^n$ around p (wlog map to some polydisc). Note $f \circ \varphi$ satisfies the max. modulus principle on U. Hence $\varphi \circ f$ constant (by ex. Sheet 1 Q1), so by bijectivity of φ f is constant on U. M is covered by finitely many charts (as compact), so repeat above for each chart. So f is constant on M.

□

Examples of Complex manifolds

0. Trivially any open subset of $\mathbb{C}^n$

1. 1-dim. complex manifold is a Riemann surface

Classification is known as the uniformilisation theorem :

Riemann sphere $\mathbb{CP}^1 \overset{\text{topol.}}{\cong} S^2$, $\mathbb{C}$, $\mathbb{C}/\mathbb{Z} \overset{\text{biholo}}{\cong} \mathbb{C}^*$,

elliptic curves $\mathbb{C}/\Lambda$ $(\overset{\text{biholo}}{\cong} S^1 \times S^1)$ where $\Lambda = \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z}$, $(\lambda_1/\lambda_2 \notin \mathbb{R})$

and Δ/Γ, $\Delta = \{|z|<1\} \subset \mathbb{C}$ and Γ subgroup of Möbius transformations of Δ properly discontinuously

More generally, the quotient construction of complex manifolds (sheet 1, Q2)

2. $\mathbb{CP}^n$ (or $\mathbb{P}^n$) Complex projective spaces

$= \{$ 1-dim subspaces in $\mathbb{C}^{n+1} \}$, $\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{CP}^n$

Notation : points in $\mathbb{CP}^n$ are written $[z_0 : \dots : z_n]$ $(= (\lambda z_0, \dots \lambda z_n) \; \forall \lambda \in \mathbb{C}\setminus\{0\}))$

With quotient topology: Hausdorff, second countable, compact.

Coord charts: $U_i = \{ [z_0 : \dots : z_n] \mid z_i \neq 0 \}$ for $i = 0, \dots, n$.

$\varphi_i ([z_0 : \dots : z_n]) = \left(\frac{z_0}{z_i}, \dots, \hat{i}, \dots, \frac{z_n}{z_i} \right) \in \mathbb{C}^n$

and $\Rightarrow$ $(j>i)$ $\varphi_j \circ \varphi_i^{-1} = \left(\frac{w_1}{w_j}, \dots, \frac{w_{i-1}}{w_j}, \frac{1}{w_j}, \frac{w_{i+1}}{w_j}, \dots, \frac{w_{j-1}}{w_j}, \frac{w_{j+1}}{w_j}, \dots, \frac{w_n}{w_j} \right)$ check.

This is a very important manifold:

(1) Compact complex manifolds <u>never</u> embed (holo) in $\mathbb{C}^n$, but some can embed in $\mathbb{CP}^n$.

These are called projective manifolds.

proof of (1) : if M is a compact complex manifold with $\iota : M \hookrightarrow \mathbb{C}^n$ a holomorphic embedding. Then $\forall \, k = 1, \dots, n$, write $\pi_k : \mathbb{C}^n \to \mathbb{C}$, $z \mapsto z_k$ the coord projection (which is holo). So $\pi_k \circ \iota : M \to \mathbb{C}$ is holomorphic on a compact complex manifold. Then $\pi_k \circ \iota$ is constant $\forall \, k$. Hence $\Rightarrow$ $\iota(M)$ is one point ↯.

□

Easy example: making S^2 into a complex 1-dim. manifold $\mathbb{CP}^1$. Note $S^2 = \{ x^2 + y^2 + z^2 = 1 \} \subset \mathbb{R}^3$. Define

$$f : (x,y,z) \in S^2 \longrightarrow \begin{cases} \left[\frac{x+iy}{1-z} : 1 \right] & \text{if } z \neq 1 \\ \left[1 : \frac{x-iy}{1+z} \right] & \text{if } z \neq -1 \end{cases} \in \mathbb{CP}^1$$

Check $f : S^2 \to \mathbb{CP}^1$ is a diffeomorphism.

The induced charts on S^2 are stereographic projections for $(0,0,\pm 1)$.

3. **Complex Tori :** $\mathbb{C}^n/\Lambda$ where $\Lambda \cong \mathbb{Z}^{2n}$ lattice, a discrete subgroup of $\mathbb{C}^n$.

Endow with quotient topology: Hausdorff, second countable and compact.

Charts: local inverses of the quotient map

$$\pi : \underline{z} \in \mathbb{C}^n \longrightarrow \underline{z} + \Lambda \in \mathbb{C}^n/\Lambda$$

$D_i \subseteq \mathbb{C}^n$ a sufficiently small open ball s.t $\varphi_i := \pi|_{D_i} \to D_i$ can be inverted.

any transition function $\varphi_j \circ \varphi_i^{-1}(\underline{z}) = \underline{z} + \lambda_{ij}(\underline{z})$, $\underset{\text{cont.}}{\lambda_{ij}} : \underset{\text{open connected}}{V} \subset \mathbb{C}^n \to \Lambda \Rightarrow \lambda_{ij}$ constant.

4. **Hopf surface:** $H^2 = \mathbb{C}^2 \setminus \{(0,0)\} \big/ (\underline{z} \sim 2\underline{z})$ is a complex manifold according to Example sheet 1 question 2.

As a real 4-manifold, it is diffeomorphic to $S^3 \times S^1$. One can show (later) that H^2 is not projective. We can also generalize this to higher dimensions. We can define H^n for each $n \in \mathbb{N}$ as in the example sheet. H^1 is biholomorphic to an elliptic curve (1 dim. complex torus).

5. **Complex Grassmannians:** start with V an n-dimensional complex vector space. Then

$$Gr_k(V) = \{ k\text{-dim complex-linear subspaces } W \subseteq V \} , \quad k < n$$

e.g. $k=1$, then $V = \mathbb{C}^{n+1}$, then $Gr_k(V) = \mathbb{CP}^n$.

How is this a valid manifold? W can be given by some $k \times n$ complex matrix with $\text{rank}_{\mathbb{C}} = k$ (choice of basis). We may diagonalise a non-singular $k \times k$ part to obtain $k(n-k)$ "free parameters".

Remark: $Gr_k(V)$ is compact, $k(n-k)$-dimensional complex manifold. Moreover, it is also projective (Q6, Sheet 1).

Proof of compactness: $(\mathbb{C}^{k,n})^* := \{$ linearly independent k-tuples in $\mathbb{C}^n \} \subset \mathbb{C}^{k,n}$ is open. Define projection map

$$\pi : (\mathbb{C}^{k,n})^* \longrightarrow Gr_k(\mathbb{C}^n)$$

which induces the quotient topology on $Gr_k(\mathbb{C}^n)$, making π continuous. Denote $(\mathbb{C}^{k,n})^*_u$ to mean the orthonormal k-tuples (wrt Hermitian inner product). Then $(\mathbb{C}^{k,n})^*_u \subset \mathbb{C}^{nk}$ is closed and bounded. Hence $(\mathbb{C}^{n,k})^*_u$ is compact. Since $Gr_k(\mathbb{C}^n)$ is the image of a compact set under a continuous map, it is also continuous.

6. **Complex Lie groups.** $G : (g,h) \in G \times G \to gh^{-1} \in G$ holomorphic.

E.g. $GL(n, \mathbb{C})$ open in $Mat_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$.

$so(n, \mathbb{C})$ also a lie group. Proof is similar to real analogue.

N.B. $so(n, \mathbb{C})$ not compact but $so(n, \mathbb{R})$ compact.

Non e.g. $\mathcal{U}(n)$ is not a complex manifold.

2. Tangent spaces and Holomorphic tangent bundles

Let M be a complex $2n$-dim manifold. Then M is a real $2n$-dim manifold. For $p \in M$, let $z_j = x_j + iy_j$ be local complex coords around p. The x_j, y_j are our real coords.

The (real) tangent space $T_pM = \text{span}_{\mathbb{R}} \langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \rangle_{j=1,\dots,n}$.

Then set $J_p \in GL_{\mathbb{R}}(T_pM) \subset \text{End}_{\mathbb{R}}(T_pM)$ by $\frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial y_j}$, $\frac{\partial}{\partial y_j} \mapsto -\frac{\partial}{\partial x_j}$ $\forall j$ $\quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

I.e. $J_p^2 = \underset{\text{identity}}{-I}$

Consider the complexified tangent space: $T_pM \otimes_{\mathbb{R}} \mathbb{C} = \text{span}_{\mathbb{C}} \langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \rangle_{j=1,\dots,n}$.

We can think of J_p as an endomorphism $\in \text{End}_{\mathbb{C}}(T_pM \otimes \mathbb{C})$ by it's complex linear extension. Still have that $J_p^2 = -I$. Every eigenvalue of J_p must square to -1, so possible e values are $\pm i$.

Define complex subspaces

$$* \begin{cases} T_p^{1,0}M = \{ v \in T_pM \otimes \mathbb{C} : J_pv = iv \} & \text{holomorphic tangent space} \\ T_p^{0,1}M = \{ v \in T_pM \otimes \mathbb{C} : J_pv = -iv \} & \text{antiholomorphic tangent space} \end{cases}$$

We have a complex conjugation on $T_pM \otimes \mathbb{C}$ induced by $\underline{e} \otimes \lambda \mapsto \underline{e} \otimes \bar{\lambda}$ $\forall \underline{e} \in T_pM$ and $\lambda \in \mathbb{C}$ and extending linearly over reals. It's not complex linear but is real linear and invertible. The map interchanges $T_p^{1,0}M \to T_p^{0,1}M$, $T_p^{0,1}M \to T_p^{1,0}M$, since J_p has real coefficients.

Proposition: (i) $\dim_{\mathbb{C}} T_p^{1,0}M = \dim_{\mathbb{C}} T_p^{0,1}M = n$ $\quad (\dim(M) = n)$

(ii) J_p and hence $*$ is defined independent of choice of coordinates. Moreover, J_p, $p \in M$ defines a smooth section of $\text{End}(TM)$. Call $J \in \text{End}(TM)$, $J(p) := J_p$.

right now just think of as variables

proof: (i) Consider a change of basis $\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}$ (e.g. $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$, $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$)

direct calculation $\Rightarrow$ $T_p^{1,0}M = \{ v - iJv \mid v \in T_pM \}$ $\quad$ $T_p^{0,1}M = \{ v + iJv \mid v \in T_pM \}$

spanned by $\{\frac{\partial}{\partial z_j}\}_{j=1,\dots,n}$ $\quad$ spanned by $\{\frac{\partial}{\partial \bar{z}_j}\}_{j=1,\dots,n}$

Thus as a real vector space, T_pM is isomorphic to $T_p^{1,0}M$ and $T_p^{0,1}M$. So loosely speaking, as real v.s. $\dim_{\mathbb{R}}(T_pM) = 2n$ and $\dim_{\mathbb{R}}(T_p^{1,0}M) = \dim_{\mathbb{R}}(T_p^{0,1}M) = 2n$, so $\dim_{\mathbb{C}}(T_p^{1,0}M) = \dim_{\mathbb{C}}(T_p^{0,1}M) = n$.

(ii) Recall that real tangent vectors are equivalent to derivations $\left(\sum_i x^i \frac{\partial}{\partial x^i}\right)(f)$ for $f \subset C^\infty(M)$ acting on $C^\infty(M, \mathbb{R})$. So complex tangent vectors are respectively derivations acting on $C^\infty(M, \mathbb{C})$ by looking at the real and imaginary parts. Thus

$T^{0,1}$ = derivations vanishing precisely on holomorphic functions on M

$T^{1,0}$ = derivations vanishing precisely on antiholomorphic functions on M

$\Rightarrow$ the $\pm i$ eigenspaces are invariantly defined because the above two statements are invariant of local coords. Taking their direct sum gives $T^{1,0} \oplus T^{0,1} = TM \otimes \mathbb{C}$. Thus J is invariantly defined on all $TM \otimes \mathbb{C}$. (Smoothness in p comes from the local coord expression). ☐

Lemma: On overlaps of complex coord. nhoods with coords (z_j), (w_j), we have

$$\frac{\partial}{\partial w_k} = \sum_j \frac{\partial z_j}{\partial w_k} \frac{\partial}{\partial z_j} \quad \text{and} \quad \frac{\partial}{\partial \bar{w}_k} = \sum \frac{\partial \bar{z}_j}{\partial \bar{w}_k} \frac{\partial}{\partial \bar{z}_j}$$

Recall that for a holo $f: U \subseteq \mathbb{C}^n \to \mathbb{C}^m$, the Jacobian $J_{\mathbb{R}}(f)$ wrt. $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ is similar via $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j}\right)$ to $\begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix}$

Combining with the lemma:

Prop: every real manifold underlying a complex manifold is oriented.

proof: Indeed, $\det J_{\mathbb{R}}(f) > 0$ holds $\forall z = f(w)$ on the overlap of coord nhoods 5.59 $\square$

Define $\bigsqcup_{p \in M} T_p^{1,0} M =: T^{1,0} M$, the holomorphic tangent bundle of M

Rem: this is a complex subbundle of $TM \otimes \mathbb{C}$.

Sections of this bundle $T^{1,0}M$ act also as derivations on $C^\infty(M, \mathbb{C})$.

Dfn: a section $\xi \in \Gamma(T^{1,0}M)$ is a __holomorphic__ vector field if $\forall f \in C^\infty(M, \mathbb{C})$ holomorphic ξf is also holomorphic.

Can also define $\bigsqcup_{p \in M} T_p^{0,1} M = T^{0,1} M$ to be the antiholo tangent bundle.

Note: $f: U \subset M \to \mathbb{C}$ is holomorphic if $\xi f = 0 \;\; \forall \xi \in \Gamma(T^{0,1}M)$

Recall $J \in \Gamma(\operatorname{End} TM)$ (from prop (ii) J is well defined)

Remark: we have a standard representation of $GL(n, \mathbb{C})$ on $\mathbb{R}^{2n}$ i.e. an injective homomorphism $GL(n,\mathbb{C}) \xrightarrow{\phi} GL(2n, \mathbb{R})$ Each complex entry $a + ib$ becomes a 2×2 real matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.
Then $\phi(GL(n,\mathbb{C}))$ = subgroup of $GL(2n, \mathbb{R})$ commuting with

$$J_0 = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$$

So $\forall f$, we have that a change of complex coords gives that Cauchy-Riemann for $f \Rightarrow J_{\mathbb{R}}(f) \in \phi(GL(n,\mathbb{C}))$.

Then a holomorphic atlas of M, via J, induces a __reduction__ of the structure group of the v.b. TM from $GL(2n,\mathbb{R})$ to $GL(n,\mathbb{C})$, hence making TM into a complex v.b. ($\operatorname{rank}_{\mathbb{C}} = n$). This happens to be isomorphic to the holomorphic tangent bundle via $v \in TM \to (v - iJv) \in T^{1,0}M$.
Locally, this is induced by $a \frac{\partial}{\partial x_k} + b \frac{\partial}{\partial y_k} \mapsto 2(a + ib) \frac{\partial}{\partial z_k}$

Using $-J$ in place of J, we get an isomorphism of v.b. $TM \to T^{0,1}M$.

Recall: if $f : M \to N$ is smooth between real manifolds, then

$$d(f)_p : T_pM \to T_{f(p)}N \quad \text{linear}$$

We can construct a complex extension of this by:

$$d(f)_p : T_pM \otimes \mathbb{C} \to T_{f(p)} \otimes \mathbb{C} \qquad \text{(complex linear)}$$

Prop: For a smooth map between complex manifolds, $f : M \to N$, then the following are equivalent.

(i) f is holomorphic

(ii) $df \circ J_M = J_N \circ df$

(iii) $df(T^{1,0}M) \subseteq T^{1,0}N$.

(iv) $df(T^{0,1}M) \subseteq T^{0,1}N$.

proof: all statements are local $\Rightarrow$ wlog we may consider $f : \underset{\text{open}}{U} \subset \mathbb{C}^n \to \mathbb{C}^m$. Use real basis $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right\}$ on $\mathbb{C}^n$ $(= \mathbb{R}^{2n})$, and $\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial u_m}, \frac{\partial}{\partial v_m}\right\}$ for $\mathbb{C}^m$ $(= \mathbb{R}^{2m})$.

(i) f is holomorphic $\Leftrightarrow$ Cauchy-Riemann eqns on U hold

need to go over proof. $\Leftrightarrow$ $(df)_p$ expressed as $J_{\mathbb{R}}(f)_p$ consisting of blocks of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a,b \in \mathbb{R}$.

Now $T^{1,0}_p$ is spanned over $\mathbb{C}$ by $\underline{e}_k = \begin{pmatrix} \vdots \\ 0 \\ 1 \\ -i \\ \vdots \\ 0 \\ \vdots \end{pmatrix}$ all entries are 0 except $\begin{cases} (2k-1)^{th} = 1 \\ (2k)^{th} = -i \end{cases}$ $J_pv = iv$ n dimensional

Then $J_{\mathbb{R}}(f)\,\underline{e}_k = \underset{l=1,\dots,n}{\begin{pmatrix} a_l & -ib_l \\ b_l & -ia_l \end{pmatrix}} \begin{matrix} 2l-1 \\ 2l \end{matrix}$ which is again a vector of type $1,0$ $\Rightarrow$ (iii)

> If f is holo, then the Cauchy Riemann equations hold. We saw in the previous section that the complex Jacobian $J(f)_z$ is then similar to $J_{\mathbb{R}}(f)$. But this complex Jacobian is exactly $(df)_p$. I mean, the complex Jacobian is precisely the expression of $(df)_p$ as a matrix when considering $(df)_p$ acting on the tangent space T (or $T^{1,0}$ or $T^{0,1}$, whichever you like) which is of course a vector space. Remember from direct calculation that $T^{1,0}_p = \{v - iJv \mid v \in T_pM\}$, which is n dimensional. The relation for vectors in $T^{1,0}_p$ is equivalently $J_pv = iv$. The e_k given above are clearly linearly independent, satisfy this rule, and there are n of them, so obviously they form a basis. The last line is straight forward.

(iii) $\Leftrightarrow$ (iv) $\underset{\uparrow \text{real}}{df}$ is invariant under complex conjugation, and $(\bar{\text{conj}})$ maps $T^{1,0}M \to T^{0,1}M$, $T^{1,0}N \to T^{1,0}N$.

(iii) and (iv) $\Rightarrow$ (ii) (df) preserves the $(1,0)$ and $(0,1)$ -subspaces. But J_M, J_N acts on these by $(\pm i)$ id.

> $\left.\begin{aligned} df\, J_M(T^{1,0}M) &= df(i\,T^{1,0}M) = i\,df(T^{1,0}M) \subseteq i\,T^{1,0}N \\ J_N\, df(T^{1,0}M) &= J_N\, df(T^{1,0}M) \subseteq J_N(T^{1,0}N) = i\,T^{1,0}N \end{aligned}\right\}$ same. similarly for $(0,1)$.

(ii) $\Rightarrow$ (i) each (2×2) block $B_{kl} := \begin{pmatrix} c_{2k-1,2l-1} & c_{2k-1,2l} \\ c_{2k,2l-1} & c_{2k,2l} \end{pmatrix}$ of $(df)_p$ commutes with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ by assuming (ii)

$\Rightarrow$ $B_{kl} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for some $a,b \in \mathbb{R}$

$\Leftrightarrow$ Cauchy Riemann hold

$\Leftrightarrow$ (i).

$\square$

3. Complexified cotangent space

$T_p^*M \otimes \mathbb{C}$: maps $T_pM \otimes \mathbb{C} \to \mathbb{C}$.

$$* \begin{cases} T_p^{1,0}M = \{ v \in T_pM \otimes \mathbb{C} : J_p v = iv \} \\ T_p^{0,1}M = \{ v \in T_pM \otimes \mathbb{C} : J_p v = -iv \} \end{cases}$$

the (dual of) J acts $dx_j \mapsto -dy_j$, $dy_j \mapsto dx_j$ $\forall j = 1, \dots, n$.

We have $dz_k = dx_k + i dy_k$, $d\bar{z}_k = dx_k - i dy_k$ } note $dz_i\left(\frac{\partial}{\partial z_j}\right) = \delta^i_j$ and similarly for $d\bar{z}$.

Then dz_k generates the (+i) eigenspace, and $d\bar{z}_k$ the -i eigenspace

missing a line here

$(T_p^{1,0}M)^* = \{ \varphi \in T_p^*M \otimes \mathbb{C} : J_p^* \varphi = i\varphi \}$

$(T_p^{0,1}M)^* = \{ \varphi \in T_p^*M \otimes \mathbb{C} : J_p^* \varphi = -i\varphi \}$

Rem: $\left\{ \frac{\partial}{\partial z_k} \right\}$ spans $T^{1,0}M$, and since the holo cotangent space is the dual of the holo tangent space, we immediately get a basis for $(T^{1,0}M)^* = \{ dz_k \}$ by the above observation. The idea of spanning the (i) and (-i) eigenspaces follows from this dualisation.

Recall subbundles $(T^*M)^{1,0}$, $(T^*M)^{0,1}$ on a complex manifold M induced by J.

We can define

$$\Lambda^r (T^*M \otimes \mathbb{C}) = \bigoplus_{p+q=r} \Lambda^{p,q} (T^*M \otimes \mathbb{C})$$

where $\Lambda^{p,q}(T^*M \otimes \mathbb{C}) = \Lambda^p (T^*M)^{1,0} \wedge \Lambda^q (T^*M)^{0,1}$.

This is just the standard (complexified) wedge product.

Under complex conjugation we have that $\overline{\Lambda^{p,q}} = \Lambda^{q,p}$ $\forall q, p$.

The sections are $\Omega^{p,q}(M)$: complex differential forms of type (p,q). In local coordinates,

$$\sum_{I,J} a_{I,J} \underbrace{dz_{i_1} \wedge \dots \wedge dz_{i_p}}_{dz_I} \wedge \underbrace{d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}}_{d\bar{z}_J}$$

The induced action of J is $J\varphi = i^{p-q} \varphi$ for $\varphi \in \Omega^{p,q}$ (we're just dropping the dual J notation for J)

Notice that $J dz_j = i dz_j$ and $J d\bar{z}_j = -i d\bar{z}_j = i^{-1} d\bar{z}_j$ since $(-1) i = (i)^{-2} \cdot i = (i)^{-1}$

Note that $\Omega^{p,p}(M) \cap \underset{\text{real forms}}{\Omega^{2p}(M)} = \Omega^{p,p}_{\mathbb{R}}(M)$ real (p,p)-forms (i.e. invariant under complex conjugation)

$K_M = \Lambda^{n,0}(T^*M \otimes \mathbb{C})$, where $n = \dim_{\mathbb{C}} M$ $= \Lambda^n (T^*M)^{1,0}$, is called the canonical line bundle

Recall on a real manifold we have an exterior derivative:

$$d: \Omega^0(M) \to \Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$$

So write $d = \partial + \bar\partial$, where $\partial = \pi^{1,0} \circ d$, $\bar\partial = \pi^{0,1} \circ d$, where $\pi^{p,q} : \Omega^*(M) \to \Omega^{p,q}(M)$ projection along other components of $\oplus$.

Locally, for a complex function f, $\partial f = \sum_k \frac{\partial f}{\partial z_k} dz_k$, $\bar\partial f = \sum_k \frac{\partial f}{\partial \bar z_k} d\bar z_k$

Generally for $\alpha \in \Omega^{p,q}(M)$ (pure type, only nontrivial component in one type), then define

$$\partial\alpha := (\pi^{p+1,q} \circ d)\alpha$$
$$\bar\partial\alpha := (\pi^{p,q+1} \circ d)\alpha$$

For $\alpha \in \Omega^{p,0}(M)$, $\bar\partial\alpha = 0$ iff locally $\alpha = \sum_I f_I(z)\, dz_{i_1} \wedge \dots \wedge dz_{i_p}$ with all f_I holomorphic. Such a form α is then called a holomorphic p-form. Holomorphic 1-forms are sometimes called holomorphic differentials.

Lemma: On a complex manifold M

(i) $\forall \eta \in \Omega^{p,q}(M)$, $d\eta = \partial\eta + \bar\partial\eta$ (the above d is defined on $\Omega^0(M)$, we're extending to higher degrees)

(ii) $\partial^2 = 0 = \bar\partial^2$, and $\partial\bar\partial = -\bar\partial\partial$

(iii) $\bar\partial(\xi \wedge \eta) = \bar\partial\xi \wedge \eta + (-1)^{p+q}\xi \wedge \bar\partial\eta$, $\xi \in \Omega^{p,q}(M)$ (Similarly for ∂)

proof: the statements are local

(i) easy to check in local coords: $d(f\, dz_I \wedge d\bar z_J) = (\partial f + \bar\partial f)\, dz_I \wedge dz_J$ and extend by linearity. Note both sides defined are independent of choice of coords).

(ii) Clear from (i) and $d^2 = 0$. $\partial f\, dz_I \wedge dz_J + \bar\partial f\, dz_I \wedge dz_J$

$$0 = d(\partial f\, dz_I \wedge dz_J + \bar\partial f\, dz_I \wedge dz_J)$$
$$= \partial^2 f\, dz_I \wedge dz_J + \bar\partial\partial f\, dz_I \wedge dz_J + \partial\bar\partial f\, dz_I \wedge dz_J + \partial^2 f\, dz_I \wedge dz_J \quad (*)$$

Then the idea is to compare the terms $\dots dz_I \wedge dz_J$, and notice that the ones that have ∂^2 and $\bar\partial^2$ coefficients have no way to cancel except for when ∂^2 and $\bar\partial^2 = 0$. The terms $\bar\partial\partial f\, dz_I \wedge dz_J$ and $\partial\bar\partial f\, dz_I \wedge dz_J$ have the same monomials (this takes a little bit of a check with antisymmetry of wedge) and so for $(*)$ to vanish we need $\partial\bar\partial = -\bar\partial\partial$.

(iii) wlog let $\eta \in \Omega^{p',q'}(M)$ be pure type. Take $(p+p'+1, q+q')$ and $(p+p', q+q'+1)$ components of $d(\xi \wedge \eta)$. Then extend complex-linearly to any η

say $\eta = \Omega^{p',q'}(M)$, $\eta = f\, dz_1 \wedge \dots \wedge dz_{p'} \wedge d\bar z_1 \wedge \dots \wedge d\bar z_{q'} = f\, dz_I \wedge dz_J$.
Take $\xi \wedge \eta$, and then apply d:

$$d(\xi \wedge \eta) = d(\xi \wedge f\, dz_I \wedge d\bar z_J)$$
$$= d\xi \wedge (f\, dz_I \wedge d\bar z_J) + (-1)^{p'+q'}\xi \wedge d(f\, dz_I \wedge d\bar z_J) \quad \text{(standard result)}$$
$\Rightarrow$ result follows considering ∂ and $\bar\partial$.

□

Corollary: $d(\Omega^{p,q}(M)) = \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$ (from (i))

Also (iii) and (i) continue to hold without the pure type assumption by complex-linearity.

It is sometimes convenient to (equivalently) replace $\partial, \bar\partial$ by d and d^c

$d = \partial + \bar\partial$, and $d^c = i(\bar\partial - \partial)$.

Both act on real differential forms. So $\partial = \frac{1}{2}(d + id^c)$, and $\bar\partial = \frac{1}{2}(d - id^c)$. Also $(d^c)^c = 0$, and

$$dd^c = -d^c d = 2i\partial\bar\partial$$

Recall pull-back of differential forms by a smooth map $f: M \to N$ is $f^*: \Omega^1(N)^{\mathbb{C}} \to \Omega^1(M)^{\mathbb{C}}$. by $\langle f^*\alpha, X\rangle = \langle \alpha, (df)(X)\rangle$ $\forall$ vector fields X

If f is holomorphic and $\alpha \in \Omega^{1,0}(M)$ (resp. $\Omega^{0,1}(M)$) (i.e. $J\alpha = i\alpha$), then

$$\begin{aligned} \langle J f^*\alpha, X\rangle &= \langle f^*\alpha, JX\rangle && \alpha \text{ is complex linear ?} \\ &= \langle \alpha, (df) JX\rangle && df \circ J = J \circ df \text{ from our previous lemma} \\ &= \langle \alpha, J(df)X\rangle \\ &= \langle J\alpha, (df)X\rangle \\ &= \langle i\alpha, (df)X\rangle \\ &= \langle i f^*\alpha, X\rangle && \forall X \end{aligned}$$

Thus $f^*\alpha \in \Omega^{1,0}(M)$, and similar for $\Omega^{0,1}(M)$. Further, since $f^*(\xi \wedge \eta) = f^*\xi \wedge f^*\eta$, (naturality)

Proposition: the pullback by a holo map preserves the type decomposition.

Further, $f^* \circ d = d \circ f$ $\forall$ smooth f, if f is holomorphic, then $\forall \eta \in \Omega^{p,q}(M)$,

$$(\bar\partial \circ f^*)\eta = (\pi^{p,q+1} \circ d \circ f^*)\eta = \pi^{p,q+1} \circ f^* \circ d\eta \underset{\substack{\uparrow \\ f \text{ is holo (from proposition preserves type)}}}{=} f^* \circ \pi^{p,q+1} \circ d\eta = f^* \circ \bar\partial\eta$$

We obtain the following proposition

Proposition: $\bar\partial \circ f^* = f^* \circ \bar\partial$ and similarly $\partial \circ f^* = f^* \circ \partial$ whenever f is holomorphic.

Definition: Dolbeault Cohomology

$$H^{p,q}(M) = \frac{\{\operatorname{Ker} \bar\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)\}}{\{\operatorname{Im} \bar\partial : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M)\}}$$

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Corollary: if $f: M \to N$ is holomorphic, then $f^*: H^{p,q}(N) \to H^{p,q}(M)$ is a well-defined, complex-linear map. Moreover, if f is biholomorphic, then $f^*: H^{p,q}(N) \xrightarrow{\sim} H^{p,q}(M)$ is an isomorphism.

Remarks ① not true in general that $\bigoplus_{p+q=r} H^{p,q}(M) = H_{dR}^r(M)^{\mathbb{C}}$

② $H^{p,q}(M)$ are not topological invariants.

notation: for $S \subseteq \mathbb{R}^n$ or $\mathbb{C}^n$, and $f: \quad \mathbb{R}$, $f \in C^\infty(S)$ means $\exists$ open $U \supset S$, smooth (C^∞) $F: U \to \mathbb{R}$ such that $f = F|_S$.

$\bar{\partial}$ - Poincaré lemma in one variable

Let $D = \{ z \in \mathbb{C} : |z-a| < r \}$, $g \in C^\infty(\bar{D})$, where $\bar{D}$ is the closed disc. Then we can define a smooth function

$$f(z) := \frac{1}{2\pi i} \int_D \frac{g(w)}{w-z} \, dw \wedge d\bar{w} \quad \in \quad C^\infty(D) \quad \text{satisfying} \quad \frac{\partial f}{\partial \bar{z}} = g \quad \text{on } D.$$

To prove this, we need a lemma known as the **extended Cauchy Integral formula:**

If $F \in C^\infty\{ |z-a| \leq r \}$ and $z \in \mathbb{C}$ s.t $|z-a| < r$, then

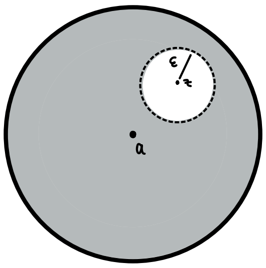
$$F(z) = \frac{1}{2\pi i} \int_{|w-a|=r} \frac{F(w)}{w-z} \, dw + \frac{1}{2\pi i} \int_{|w-a|<r} \frac{\partial F}{\partial \bar{w}}(w) \, \frac{dw \wedge d\bar{w}}{w-z}$$

proof: Stokes theorem for 1-form $\eta = \frac{1}{2\pi i} \frac{F(w)}{w-z} dw$ on $D_\varepsilon = \{ |w-a| < r \} \setminus \{ |w-z| < \varepsilon \}$

Rem: if we assume F is holo, then $\frac{\partial F}{\partial \bar{w}} = 0$ and second term vanishes (get original Cauchy Integral formula)

Then $d\eta = \frac{-1}{2\pi i} \frac{\partial F}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z}$. Calculate then that

$$\int_{D_\varepsilon} d\eta = \int_{|w-a|=r} \eta - \int_{|w-z|=\varepsilon} \eta \qquad \text{(Stokes w/ boundary)}$$



$D_\varepsilon \to D$ as $\varepsilon \downarrow 0$

Now $\int_{|w-a|=\varepsilon} \eta = \frac{1}{2\pi i} \int_0^{2\pi} F(z + \varepsilon e^{i\theta}) \, d\theta \longrightarrow F(z)$ as $\varepsilon \downarrow 0$ (tends to 0 from above)

(This just follows by using a change of coordinates $w \mapsto \theta$, $w = z + \varepsilon e^{i\theta}$.

Pole order 1 is integrable : $\left| \frac{\partial F}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} \right| = \left| \frac{\partial F}{\partial \bar{w}} \cdot \frac{2 dx \wedge dy}{\underset{r^2 = x^2+y^2}{r}} \right| \overset{}{\underset{\text{to polar coords}}{=}} \underbrace{\left| 2 \frac{\partial F}{\partial \bar{w}} dr \wedge d\theta \right|}_{\text{which is integrable}}$ } Justification that LHS makes sense.

($dw \wedge d\bar{w} = -2i \, dx \wedge dy = -2ir \, dr \wedge d\theta$)

$w = x + iy$, (r, θ) polar from (x,y).

Can take $\varepsilon \to 0$ in 2-dimensional integral.

Putting this all together (as $\varepsilon \downarrow 0$),

$$\int_D -\frac{1}{2\pi i} \frac{\partial F}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} = \int_{|w-a|=r} \frac{1}{2\pi i} \frac{F(w)}{w-z} dw - F(z)$$

$$\Leftrightarrow \quad F(z) = \frac{1}{2\pi i} \int_{|w-a|=r} \frac{F(w)}{w-z} dw + \frac{1}{2\pi i} \int_{|w-a|<r} \frac{\partial F}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z}, \quad \text{as required.}$$

☐

Proof of 1-dim Poincaré

Let $z_0 \in D$, and Choose $D_0 := \{|z - z_0| < 2\varepsilon\} \subset D$ $(\bar{D_0} \subset D)$

$g(z) = g_1(z) + g_2(z)$ both smooth s.t $g_1|_{\{|z-z_0| \geq 2\varepsilon\}} \equiv 0$ and $g_2|_{\{|z-z_0| \leq \varepsilon\}} \equiv 0$

Let $f_2(z) = \frac{1}{2\pi i}\int_D \frac{g_2(w)}{w-z}\, dw \wedge d\bar{w}$. The function we're integrating is smooth and bounded, for z near z_0 so integral is well defined - <u>not</u> an improper integral.

Then $\frac{\partial f_2}{\partial \bar{z}}(z) = \frac{1}{2\pi i}\int_D \underbrace{\frac{\partial}{\partial \bar{z}}\left(\frac{g_2(w)}{w-z}\right)}_{\text{function vanishes}} dw \wedge d\bar{w}$ function is smooth and bounded, and on a bounded domain.

$= 0$

So just need to consider g_1 for the proposed definition of f.

g_1 has compact support, so

$$f_1(z) := \frac{1}{2\pi i}\int_D \frac{g_1(w)}{w-z}\, dw \wedge d\bar{w} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(w)}{w-z}\, dw \wedge d\bar{w}$$ extend by 0

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g(u+z)}{u}\, du \wedge d\bar{u}$$ $u = w - z$

$$= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g_1(z + re^{i\theta})}{e^{i\theta}}\, dr \wedge d\theta$$

$\Rightarrow$ well defined and smooth in z.

Consider then

$$\frac{\partial f_1}{\partial \bar{z}} = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial g_1}{\partial \bar{z}}(z + re^{i\theta})e^{-i\theta}\, dr \wedge d\theta = \underbrace{\frac{1}{2\pi i}\int_D \frac{\partial g_1}{\partial \bar{w}}(w)\, \frac{dw \wedge d\bar{w}}{w-z}}_{\text{back to } w \text{ variable}}$$

From Lemma,

$$g_1(z) = \frac{1}{2\pi i}\int_{|w-a|=r} \underset{0}{\frac{g_1(\cancel{w})\, dw}{w-z}} + \frac{1}{2\pi i}\int_{|w-a|<r} \frac{\partial g_1}{\partial \bar{w}}\, \frac{dw \wedge d\bar{w}}{w-z}$$

g_1 vanishes on this contour

So in sum we have

$$\frac{\partial f_1}{\partial \bar{z}}(z_0) = g_1(z_0) = g(z_0), \text{ and } \frac{\partial f}{\partial \bar{z}}(z_0) = \frac{\partial f_1}{\partial \bar{z}}(z_0) \quad \text{as} \quad \frac{\partial f_2}{\partial \bar{z}} \equiv 0 \text{ near } z_0$$

□

(General dimensional) $\bar\partial$ - Poincaré Lemma

Let $D = \{|z_1 - a_1| < r_1\} \times \cdots \{|z_m - a_m| < r_m\} \subseteq \mathbb{C}^m$ a polydisc, with possibly some $r_k = \infty$.

Then $H^{p,q}(D) = 0$ for $q \geqslant 1$.

proof: Let $\varphi \in \Omega^{p,q}(D)$ be closed, $\bar\partial\varphi = 0$. Without loss of generality, $p = 0$ as we can always write $\varphi = \underset{(0,q)}{\eta} \wedge \underset{(p,0)}{dz_J}$. And $\bar\partial\varphi = \bar\partial\eta \wedge dz_J$, which vanishes $\Leftrightarrow$ $\bar\partial\eta = 0$. So if we can prove it for $p=0$ it extends immediately to all p.

So let $\varphi \in \Omega^{0,q}(D)$.

Claim: $\exists\ \psi \in \Omega^{0,q-1}(D_0)$ s.t $\bar\partial\psi = \varphi$ on D_0, where D_0 is a smaller polydisc with radii $\varepsilon_k < r_k$, $k=1,\dots,n$.

Proceed by "integrating" $d\bar z_n$, then $d\bar z_{n-1}, \dots$.

Suppose that only $d\bar z_1, \dots, d\bar z_k$ occur in φ. Then we can write $\varphi = d\bar z_k \wedge \varphi_1 + \varphi_2$ for some unique φ_1, φ that do not contain $d\bar z_k$. Since φ is $\bar\partial$-closed, $\Rightarrow$ for $\varphi_1 = \sum_I \varphi_I\, d\bar z_I$, $I \subseteq \{1,\dots,k\}$, we have

$$\frac{\partial\varphi_I}{\partial\bar z_\ell} = 0 \quad \forall\ \ell > k$$

Set $\eta_I := \displaystyle\int_{|w_k - a_k| \leq \varepsilon_k} \varphi(\dots, w_k, \dots)\, \frac{dw \wedge d\bar w_k}{w_k - z_k}$. Then

$$\frac{\partial\eta_I}{\partial\bar z_k} = \varphi_I$$ by Cauchy Integral formula.

But $\dfrac{\partial\eta_k}{\partial\bar z_k} = 0 \quad \forall\ (\ell > k)$ as $\dfrac{\partial\varphi_1}{\partial\bar z_k} = 0$

$\Rightarrow \quad \varphi - \bar\partial\left(\sum \eta_I\, d\bar z_I\right) = \varphi_2$ with no $d\bar z_k$ occuring.

Repeating in each variable, we obtain ψ.

NB. we needed to reduce D to D_0. To now solve the $\bar\partial$ equation on <u>all</u> of D, take $\varepsilon_k^{(n)} \uparrow r_k$ as $n \to \infty$ $\forall\ k = 1,\dots,m$. Then $\exists\, \psi_n \in \Omega^{0,q-1}(D)$ s.t $\bar\partial\psi_n = \varphi$ on D_n polydisc with $\varepsilon_k^{(n)}$ $\left(\bigcup_{n=0}^{\infty} D_n = D\right)$.

Claim: ψ_n will converge as $n \to \infty$.

pf: Induct on q: assume true for $0,\dots,(q-1)$-form φ, where $q \geqslant 2$. Then $\exists\,\alpha$ such that

$$\bar\partial\alpha = \varphi \text{ on } D_{n+1} \qquad \Rightarrow \bar\partial(\alpha - \psi_n) = 0 \text{ on } D_n$$ by inductive assumption

$\Rightarrow \exists\beta \in \Omega^{0,q-2}(D)$ such that $\bar\partial\beta = \psi_n - \alpha$ on D_{n-1}. Set $\psi_{n+1} = \alpha + \bar\partial\beta$. Then $\bar\partial\psi_{n+1} = \bar\partial\alpha = \varphi$ on D_{n+1}, and $\psi_{n+1}|_{D_{n-1}} = \psi_n|_{D_{n-1}}$. So this sequence $\{\psi_n\}$ is convergent to a well defined ψ as $n \to \infty$, and $\bar\partial\psi = \varphi$ on D.

For our induction it remains to show $\bar\partial$ - Poincaré for $(0,1)$ -forms $(q=1)$: I.e. given $\forall\ \varphi \in \Omega^{0,1}(D)$ with $\bar\partial\varphi = 0$ any $\forall$ open polydisc D_0 with $\bar{D}_0 \subset D$, $\exists\ \psi_0 \in C^\infty(D)$ with $\bar\partial\psi_0 = \varphi$ on D_0. Then in fact $\exists\ \psi_0 \in C^\infty(D)$ with $\bar\partial\psi_0 = \varphi$ on D.

As before, take same sequence D_n of polydiscs. $\exists\ \psi_n \in C^\infty(D)$ s.t $\bar\partial\psi_n = \varphi$ on D_n, and $\exists\ \alpha \in C^\infty(D)$ such that $\bar\partial\alpha = \varphi$ on D_{n+1}. Then $\psi_n - \alpha$ is a holomorphic function on D_n since it satisfies the CR equations. Thus it is represented by a power series on D_n converging uniformly on $\overline{D_{n-1}}$ (on any compact subset). Hence $\exists$ partial sum (a holo polynomial) β such that

$$\sup_{\overline{D_{n-1}}} |(\psi_n - \alpha) - \beta| < \frac{1}{2^n}$$

Set $\psi_{n+1} = \alpha + \beta \Rightarrow \bar\partial\psi_{n+1} = \bar\partial(\alpha+\beta) \underset{\beta\ \text{holo}}{=} \bar\partial\alpha = \varphi$ on D_{n+1}

Moreover, $\psi_{n+1} - \psi_n$ is holo on D_n with $\sup_{\overline{D_{n-1}}} |\psi_{n+1} - \psi_n| < \frac{1}{2^n}$, so we obtain a sequence $(\psi_n)_{n=0}^\infty$ in $C^\infty(D)$ with uniform convergence $\psi_n \to \psi$ $(n \to \infty)$ on compact subsets of D. Therefore (for all fixed n)

$$\lim_{\ell\to\infty} (\psi_\ell - \psi_n) \text{ is holomorphic on } D_{n-1}$$

(uniform limit of holo functions). and $\bar\partial\psi = \varphi$ on D (because its true $\forall n$). $\square$

What about the remaining groups?

Rem: $H^{p,0}(\mathbb{C}^n) = \{$space of all holomorphic p-forms$\}$ is infinite dim

$H^{0,0}(M) \cong \mathbb{C}$ for any compact complex manifold M since this is the space of holomorphic functions on M, and any holomorphic function on a <u>compact</u> manifold is constant.
(Shall later see $\dim H^{p,0}(M) < \infty$ for compact M if Kähler)

Almost Complex Manifolds

Definition: a smooth real manifold M is called an almost complex manifold if $\exists\ J \in \Gamma(\text{End}\ TM)$ with $J^2 = -1$. Such a J is called an almost complex structure on M.

Lemma: (from linear algebra). Let $J \in \text{End}(\mathbb{R}^m)$, $J^2 = -1$. Then 0) $J \in GL(m,\mathbb{R})$, and 1) $m = 2n$, and 3) $\{A \in GL(m,\mathbb{R}) : AJA^{-1} = J\} \cong GL(n,\mathbb{C})$.

Take $[S] \in GL(2n,\mathbb{R})/GL(n,\mathbb{C}) \longmapsto SJ_0S^{-1} = \{J \in \text{End}(\mathbb{R}^n)\ ;\ J^2 = -1\}$

where J_0 = block diag matrix with blocks $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Sketch-proof of Lemma:

$\forall\ v \neq 0$, v and Jv are linearly independent. Can get a basis from them of the form $e_1, Je_1, e_2, Je_2, \dots, e_n, Je_n$, an even number. Then $J = J_0$ in this basis.

Corollary: an almost complex structure is equivalent to a $GL(n, \mathbb{C})$ - structure on M. Thus every almost complex manifold is even dimensional and has a canonical orientation.

mean J^* implicitly.

We can extend $e_1, Je_1, \dots, e_n, Je_n$ to a local frame field around $p \in M$. Let $e_1^*, Je_1^*, \dots, e_n^*, Je_n^*$ the dual coframe field. Then $\varepsilon = Je_1^* \wedge e_1^* \wedge \dots \wedge Je_n^* \wedge e_n^*$

E.g. if M is a cx. manifold with local coords (z_j), then $\varepsilon = \frac{i^n}{2^n} dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n$,

or in real $= dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n$

$$\left(\text{Recall} \quad \begin{array}{l} J(dx) = -dy \\ J(dy) = -dx \end{array} \right)$$

J^*.

Remark: $(-J)$ is another almost complex structure, giving the same orientation if n is even
opp. orientation if n is even
$(\dim_{\mathbb{R}} M = 2n)$

Definition: The torsion of an almost complex structure J is a tensor $N_J \in \Gamma(\mathrm{Hom}(\Lambda^2 TM, TM))$

$$N_J(X,Y) = 2([JX, JY] - [X,Y] - J[X, JY] - J[JX, Y]) \in \mathfrak{X}(M), \quad X, Y \in \mathfrak{X}(M)$$

acts on real vector fields

If $N_J = 0$, then J is called torsion-free or integrable.

Fact: N_J is $C^\infty(M)$ - linear (is an algebraic map) (direct calculation using $[fX, Y] = f[X,Y] - (Yf)X$)
So coefficients of N_J depend on J + $\underbrace{1^{st}}_{?}$ derivative

05/02/2022

Remark: N_J is antisymmetric $N_J(X,Y) = -N_J(Y,X)$ and $N(fX, Y) = fN(X,Y)$ $\forall f \in C^\infty(M)$

in local coordinates: $N_J(\partial_i, \partial_j) = \sum_k N_{ij}^k \partial_k$, where $\partial_i := \frac{\partial}{\partial u_i}$, $\{u_i\}$ real local coordinates.

Newlander - Nirenberg Theorem:
An almost complex structure J on M arises from an atlas of local complex coords iff $N_J \equiv 0$ (J is torsion-free).

Remarks on the proof:

"⇒" is easy: Let $z_\alpha = x_\alpha + i y_\alpha$ be local complex coordinates. Then consider $\frac{\partial}{\partial x_\alpha}$, $\frac{\partial}{\partial y_\alpha}$, $J\frac{\partial}{\partial x_\alpha}$ and $J\frac{\partial}{\partial y_\alpha}$ have constant coefficients, and in particular their Lie bracket $[\cdot,\cdot]$ vanishes. Hence $N_J \equiv 0$, as required.

"⇐" is difficult: can be read in Kobayashi and Nomizu, but quite involved, for smooth, real analytic manifolds.

An almost complex structure J suffices for defining $T^{1,0}M$, $T^{0,1}M$ and $\Lambda^{p,q}(T^*M)^{\mathbb{C}}$. Hence $\partial, \bar{\partial}$ on $\Omega^{p,q}$ also make sense on any almost complex manifold.

However, there is a difference between almost complex manifolds and complex manifold.

Proposition: if M is an almost complex manifold, then

$$d(\Omega^{p,q}(M)) \subseteq \Omega^{p-1,q+2}(M) \oplus \Omega^{p,q+1}(M) \oplus \Omega^{p+1,q}(M) \oplus \Omega^{p+2,q-1}(M)$$

proof: obviously, $\left.\begin{array}{l} d\Omega^{0,1} \subset \Omega^{0,2} + \Omega^{1,1} + \Omega^{2,0} \\ d\Omega^{1,0} \subset \Omega^{0,2} + \Omega^{1,1} + \Omega^{2,0} \end{array}\right\} (*)$

an arbitrary (p,q)-form can be written as $\sum_{i=1}^{N} \varepsilon_1^{(i)} \wedge \dots \wedge \varepsilon_{p+q}^{(i)}$ where each $\varepsilon_j^{(i)} \in \Omega^{1,0}$ or $\Omega^{0,1}$. We can apply the product rule and $(*)$.

$\square$

Theorem : For M an almost complex manifold, the following are equivalent:

(a) $z, W \in \Gamma(T^{1,0}(M))$. Then $[z, W] \in \Gamma(T^{1,0}(M))$

(b) $z, W \in \Gamma(T^{0,1}(M))$. Then $[z, W] \in \Gamma(T^{0,1}(M))$

(c) $\left\{\begin{array}{l} d(\Omega^{1,0}(M)) \subset \Omega^{1,1}(M) \oplus \Omega^{2,0}(M) \\ d(\Omega^{0,1}(M)) \subset \Omega^{0,2}(M) \oplus \Omega^{1,1}(M) \end{array}\right.$

(d) $d(\Omega^{p,q}(M)) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$

(e) $N_J \equiv 0$ (J is integrable)

proof:

(a) $\Leftrightarrow$ (b): apply complex conjugation : since Lie bracket is a real operator, it commutes with complex conjugation. That is, $\overline{[z,W]} = [\bar{z}, \bar{W}]$, and $z \in T^{0,1} \Leftrightarrow \bar{z} \in T^{1,0}$

(a) or (b) $\Rightarrow$ (c) : let ω be any 1-form. Then

$$d\omega(z,W) = z\omega(W) - W\omega(z) - \omega([z,W]) \qquad (\dagger)$$

If $\omega \in \Omega^{1,0}(M)$, and $z, W \in T^{0,1}$, then $\omega(z) = \omega(W) = \omega(\overset{\overset{T^{1,0}}{\in}}{[z,W]}) = 0$. Hence RHS of $\dagger$ vanishes. Hence $d\omega \equiv 0$ on $T^{0,1}$, which says precisely that ω has no $\Omega^{0,2}$ components. So (c) holds. Similarly for the other case (can also use complex conjugation)

(c) $\Rightarrow$ (a) : use $(\dagger)$. Suppose $z, W \in T^{1,0}$, and $\omega \in \Omega^{0,1}$. Then by assumption, $d\omega$ is spanned by $(0,2)$ and $(1,1)$-forms, no $(2,0)$-form component. So LHS of $(\dagger) \equiv 0$ (on $T^{1,0}$). We also know $\omega(W) = \omega(z) = 0$, $\Rightarrow \omega([z,W]) = 0$. But then $[z,W]$ cannot have any $T^{0,1}$ component. So $[z,W] \in T^{1,0}$. Similarly (c) $\Rightarrow$ (b).

(c) $\Rightarrow$ (d): same calculation as in Proposition, i.e. product rule

$$\Rightarrow d(\varepsilon_1 \wedge \dots \wedge \varepsilon_{p+q}) = \sum_K \varepsilon_1 \wedge \dots \wedge (-1)^K d\varepsilon_K \wedge \dots \wedge \varepsilon_{p+q}$$

Come back to this.

(suffices to work locally)

(d) $\Rightarrow$ (c) : (c) is a special case of (d) (trivial)

(a) $\Leftrightarrow$ (e) : a general (1,0) - vector field is $X - iJX$ for some real vector field X. Define $z := [X - iJX, Y - iJY]$. By linearity, expand: $z = -[JX, JY] + [X, \overset{?}{\underline{JY}}] + iJJ[X, JY] + i\overset{J^2=-1}{JJ}[JX, Y]$
Applying J to both sides and multiplying by i, direct calculation shows $2(z + iJz) = -N_J(X,Y) - iJN_J(X,Y)$
Now LHS $= 0$ iff z is of type $(1,0)$, and RHS $= 0$ $\Leftrightarrow$ real and imaginary parts are both 0 (since X and Y are real vector fields). Hence RHS $= 0$ iff $N_J \equiv 0$.

Can then read off (a) $\Leftrightarrow$ (e), completing the proof. ☐

Remarks :

- existence of J is a topological question (about the endomorphism bundle). This question is largely understood.
- integrability of J – nonlinear P.D.E. (more difficult question).
- easy special case: real surfaces ($\dim_{\mathbb{R}} M = 2$) By dimension reasons, no $(0,2)$ - forms, $\Rightarrow$ J is always integrable. (using statement (c)).

Submanifolds and Subvarieties

recall: $Y \subset X$ is a (embedded) (C^∞) submanifold of a manifold X means the inclusion $\iota : Y \to X$ is smooth with $(d\iota)_y : T_yY \to T_yX$ injective $\forall y \in Y$, and ι is a homeo onto its image.

Then (and only then) locally Y around $y \in Y$ is the inverse image (level set) of a regular value.

Definition: let X be a complex n-manifold, $Y \subset X$ a smooth submanifold of even dimension $\dim_{\mathbb{R}} Y = 2k$. Then we say Y is a k-dim. complex submanifold iff $\forall y \in Y$

$$* \left\{ \begin{array}{l} \exists \text{ complex coordinate chart } \varphi_y : U_y \subseteq X \to \mathbb{C}^n, \ y \in U_y, \text{ such that } \varphi_y(U_y \cap Y) = \varphi_y(U_y) \cap \mathbb{C}^k, \text{ where} \\ \mathbb{C}^k \subset \mathbb{C}^n = \{ z \in \mathbb{C}^n : z_k = z_{k+1} = \dots = z_n = 0 \} \end{array} \right.$$

Remarks:

- Thus Y is a complex k-dimensional manifold with holo. atlas $\{ (U_y \cap Y, \varphi_y|) \}_{y \in Y}$.
- $\mathrm{codim}_{\mathbb{C}} Y/X = \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y = n - k$.
- $*$ is equivalent to: $\forall y \in Y$, $\exists$ holo $F : \underset{\substack{\text{open} \\ y \in W_y \\ \text{coords } (w_i)}}{W_y} \subset X \to \mathbb{C}^{n-k}$ such that $\mathrm{rk}_{\mathbb{C}} \overset{J(F)}{\left(\frac{\partial F_i}{\partial w_j} \right)} = n - k$ on W_y, and

  $F^{-1}(0) = Y \cap W_y$ (Inverse mapping thm in complex variable)

- Then $\iota : Y \hookrightarrow X$ is a holomorphic map. $\Leftrightarrow$ Equivalently, $T_yY \subset T_yX$ is a complex vector subspace.
  inclusion map commutes with action of J
  So $\Leftrightarrow$ $T_y^{1,0}Y \subset T_y^{1,0}X$ (by previous theorem).
  $$\underset{T_yY}{\underset{\cap}{e}} \overset{\sim}{\longmapsto} \underset{T_y^{1,0}Y}{\underset{\cap}{\langle e - iJe \rangle}}$$
- Recall : if $X = \mathbb{CP}^n$ and Y compact, then Y is a projective manifold.

Definition: $Y \subset X$ is called an analytic subvariety if $Y \subset X$ is a closed subset and $\forall p \in Y$ $\exists$ nbhood $U_p \subset X$ such that $U_p \cap X = f^{-1}(0)$ for some holo $f : U_p \to \mathbb{C}^m$

p is a smooth point of Y if $\exists$ such f with $\text{rank}_{\mathbb{C}} J(f)_p = m$, i.e. $(df)_p$ is surjective. Otherwise p is called a singular point.

Define: Singular locus $:= Y^s = \{$ all the singular points in $Y\}$. If $Y^s = \emptyset$, then we say Y is smooth / nonsingular. By implicit function theorem, every connected component of $Y^* := Y \setminus Y^s$ is a complex manifold

Y is said to be irreducible if $Y \neq Y_1 \cup Y_2$ for two proper subvarieties $Y_1, Y_2 \neq Y$. We can show (sheet 2), Y irreducible $\Rightarrow$ Y^* is connected. Suppose Y is irreducible. Then $\text{codim}\, Y/X := \text{codim}\, Y^*/X$

Fact: Y^s is itself a subvariety, and $\text{codim}\, Y^s/X > \text{codim}\, Y/X$. We can check weaker statements:

- $Y^* \neq \emptyset$ and is dense open in Y
- Y^s is contained in a subvariety of X. This subvariety does not contain Y.

If $\text{codim}_{\mathbb{C}}\, Y/X = 1$, then we will call Y a hypersurface.

2. Holomorphic Geometry

2.1 Holomorphic Vector bundles

Let X be a complex manifold.

Definition: a **holomorphic vector bundle** of (complex) rank k over a base X is a complex manifold E, the total space, with a holomorphic submersion $\pi : E \to X$ ($d\pi$ is surjective) onto X such that $\forall x \in X$, the fibre $\pi^{-1}(x)$ is a k-dimensional complex vector space, and $\forall y \in X$, $\exists$ nbhood $\mathcal{U}$ of y and a biholomorphic ϕ_u called a **holomorphic local trivialisation** s.t. the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(\mathcal{U}) \arrow[r, "\phi_u"] \arrow[d, "\pi"'] & \mathcal{U} \times \mathbb{C}^k \arrow[d, "pr_1"] \\
\mathcal{U} \arrow[r, "="] & \mathcal{U}
\end{tikzcd}$$

We also ask that $\phi_u|_{E_x = \pi^{-1}(x)} : E_x \to \mathbb{C}^k$ is a complex linear isomorphism for all $x \in \mathcal{U}$.

If $\mathcal{U}_\alpha, \mathcal{U}_\beta$ are overlapping trivialising nbhoods, ϕ_α, ϕ_β holomorphic local trivialisations, then

$$\phi_\beta \circ \phi_\alpha^{-1}(x, v) = (x, \psi_{\beta\alpha}(x)v)$$

for some $\psi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(k, \mathbb{C})$ holomorphic. It also makes sense to speak of holomorphic local (/global) sections of E: $s : \mathcal{U} \subset X \to E$ holomorphic and s.t $\pi \circ s = id_{\mathcal{U}}$.

Properties:

(sheet 2) if E and $\hat{E}$ are two holomorphic v.b, then $E \oplus \hat{E}$, $E \otimes \tilde{E}$, $\Lambda^r E$, $\begin{array}{c}\mathrm{End}(E) \\ \shortparallel \\ E \otimes E^* \end{array}$, E^* are all complex. v.bs.

exterior power

$$\det E = \Lambda^{rk E} E$$

Remark: $\{ \mathcal{U}_\alpha, \psi_{\beta\alpha} : \alpha, \beta \in A \}$ determines the holo. v.b. E up to isomorphism, i.e. two holo v.b E and $\tilde{E}$ are isomorphic iff $\exists$ biholo $F : E \to \tilde{E}$ such that

$$\begin{tikzcd}
E \arrow[rr, "F"] \arrow[dr, "\pi_E"'] & & \tilde{E} \arrow[dl, "\pi_{\tilde{E}}"] \\
& X &
\end{tikzcd}$$

is a commutative diagram, and $F|_{E_x}$ is a $\mathbb{C}$-linear isomorphism.

Fix notation: X complex manifold, E holo. v.b. over X.

The pullback of E via holomorphic map $f : Y \to X$ is a vector bundle f^*E over Y so that $\exists$ $\mathcal{F}$ holomorphic map with commutative diagram

$$\begin{tikzcd}
f^*E \arrow[r, "\mathcal{F}"] \arrow[d, "\pi_Y"'] & E \arrow[d, "\pi_X"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

The map $\mathcal{F}$ is given in each holo. trivialisation over $U \subset X$ say by

$$(b,v) \in f^{-1}(U) \times \mathbb{C}^k \longrightarrow (f(b), v) \in U \times \mathbb{C}^k$$

The transition functions of f^*E are $\psi_{\beta\alpha} \circ f$ for all transition functions $\psi_{\beta\alpha}$ of E. These are holo since $\psi_{\beta\alpha}$, f are holo + they satisfy the cocycle conditions. Thus f^*E is a well-defined holo v.b.

Examples:

1) $T^{1,0}X$, $(T^*X)^{1,0}$, $\Lambda^p(T^*X)^{1,0}$, $\overset{?}{K_X}$ are holo v.b., transition functions are compositions of complex Jacobians for local coords with holo functions (in fact algebraic).

2) $Y \subset X$ complex submanifold, then inclusion $\iota: Y \hookrightarrow X$ is holo, and $\iota^*E \to Y$ is a holo vector bundle – the restriction $E|_Y$

Shall mostly consider holomorphic line bundles ($\text{rank}_{\mathbb{C}} = 1$).

Proposition / Definition: holomorphic line bundles over X form an abelian group. The operation is $\otimes$, and the group is called the Piccard group, denoted $\text{Pic}(X)$.

proof: Let L be a holo line bundle with transition functions $f_{\beta\alpha}$, and $\tilde{L}$ with $\tilde{f}_{\beta\alpha}$. Then $L \otimes \tilde{L}$ has the transition functions $f_{\beta\alpha} \cdot \tilde{f}_{\beta\alpha}$ (pointwise mult.) Commutativity since they map $\to GL(1,\mathbb{C}) = \mathbb{C}^*$. The inverse of L is L^*: noting $z \in X$, f trans. funct, then $f(z): V \to W$ is a complex linear map. Consider the dual map $f(z)^*: W^* \to V^*$ given by $f(z)^T$ wrt. the dual bases. If V and W have same dimension, then this is a complex isomorphism. Take $(f(z)^T)^{-1}: V^* \to W^*$. Rank $(f(z)^T) = 1 \Rightarrow f(z) = f(z)^T$, so for complex line bundles $(f(z)^T)^{-1} = (f(z))^{-1}$. Then transition functions of L^* are exactly $f_{\beta\alpha}^{-1}$. Identical $\overset{?}{\text{element}}$ in $X \times \mathbb{C}$, the trivial product b.

☐

Corollary: if $f: Y \to X$ holo, then $f^*: \text{Pic}(X) \to \text{Pic}(Y)$ is a group homomorphism.

Example: the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^n$. Start with

$$(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{\pi} [z_0 : \dots : z_n] \in \mathbb{CP}^n.$$

We want to exhibit $\mathbb{C}^{n+1} \setminus \{0\}$ as a line bundle $E = \mathcal{O}(-1)$, minus the zero section. It suffices for working out the transition function. Start with standard coord patches

$$U_\alpha := \{ z_\alpha \neq 0 \} \subset \mathbb{CP}^n, \qquad \alpha = 0, \dots, n.$$

Define then

$$\phi_\alpha^{-1}([z_0 : \dots : z_n], w) = \left(\frac{z_0}{z_\alpha} w, \dots, \frac{z_n}{z_\alpha} w \right) = \frac{w}{z_\alpha} \vec{z} \qquad \text{(assume } w \neq 0\text{)}$$

and

$$\phi_\beta(\zeta_0, \dots, \zeta_n) = \Big(\Big[\frac{\zeta_0}{\zeta_\beta} : \dots : \underbrace{1}_{\beta^{th}} : \dots : \frac{\zeta_n}{\zeta_\beta} \Big], \zeta_\beta \Big) \quad , \quad \zeta_\beta = \boxed{\frac{z_\beta}{z_\alpha}} w$$

transition function.

So $\phi_\beta \circ \phi_\alpha^{-1}(\vec{z}, w) = (\vec{z}, \psi_{\beta\alpha}(\vec{z}) w)$ with $\underline{\psi_{\beta\alpha}(\vec{z}) = \frac{z_\beta}{z_\alpha}}$ defined on $U_\alpha \cap U_\beta$.

Thus $\mathcal{O}(-1)$ is well-defined.

Define $\mathcal{O}(1) := \mathcal{O}(-1)^*$, the hyperplane bundle. Can further define

$$\mathcal{O}(n) := \mathcal{O}(1)\underbrace{\otimes \cdots \otimes}_{n \text{ times}} \mathcal{O}(1) = \mathcal{O}(n-1)\otimes\mathcal{O}(1)$$

Also $\mathcal{O}(-n) := \mathcal{O}(-n+1) \otimes \mathcal{O}(-1)$ and $\mathcal{O}(0) :=$ trivial product.

Thus $\mathbb{Z} \to \operatorname{Pic}(\mathbb{CP}^n)$ is a homomorphism. In fact, $\operatorname{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$.

Divisors

Need to borrow some facts from commutative algebra.

Local Rings

Consider $p \in X$, and define $\mathcal{O}_{X,p} := \{$ holo functions f defined on some open region $U_f \ni p\}$, the local ring at p. We identify f and $\tilde{f}$ if $f|_{U_f \cap U_{\tilde{f}}} = \tilde{f}|_{U_f \cap U_{\tilde{f}}}$. A function $f \in \mathcal{O}_{X,p}$ is called an element. f is an invertible element at $p \Leftrightarrow f(p) \neq 0$. f is an irreducible element at $p \Leftrightarrow$ if $f = uv$, then u or v is invertible (or both). f divides g if $f = ug$ for some element $u \in \mathcal{O}_{X,p}$. Finally, f and g are coprime $\Leftrightarrow \forall u$ dividing both f and g, u is invertible.

(Weak) Nullstellensatz: let f be an irreducible element at $0 \in \mathbb{C}^n$, and let h be an element vanishing on $f^{-1}(0) \cap$ (domain of h). Then f divides h.

Let $f : D \subset \mathbb{C}^n \to \mathbb{C}$ $\overset{\text{irreducible}}{\text{holo}}$, $0 \in D$, and $h|_{f^{-1}(0)} \equiv 0$ holo. Then f divides h in $\mathcal{O}_{\mathbb{C}^n,0}$. That is, $h = uf$, where $u \in \mathcal{O}_{\mathbb{C}^n,0}$. We'll write $f \mid h$ in $\mathcal{O}_{\mathbb{C}^n,0}$ for "f divides h".

Basic example: $n = 1$, then we have an isolated zero $f(0) = 0$. Irreducibility $\Rightarrow$ zero is simple (order 1). Assume $h(0) = 0$. Then write $f(z) = z f_1(z)$, $f_1(0) \neq 0$. Then $h(z) = z^d h_1(z)$, $h_1(0) \neq 0$, and $d \geq 1$. Therefore we can write $h(z) = z^{d-1} \underbrace{\frac{h_1(z)}{f_1(z)}}_{\text{holo}} f(z)$

We shall also need:

Theorem (U.F.D): $\mathcal{O}_{\mathbb{C}^n,0}$ is a unique factorization domain. I.e. $\forall f \not\equiv 0$, we have $f = f_1 \cdots f_m$ $(m \geq 1)$ where each f_j is irreducible, and unique up to an invertible element.

Proposition: Let $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$. If f and g are coprime at 0, then $\exists\, \varepsilon > 0$ such that f and g are coprime in $\mathcal{O}_{\mathbb{C}^n,z}$ whenever $|z| < \varepsilon$.

These are useful for studying hypersurfaces $Y \subset X$ (and line bundles)

Recall if $p \in Y^*$ (= smooth locus) then $\exists$ holomorphic $f : \underset{\text{open nhood of } p}{U_p} \subset X \to \mathbb{C}$ such that $Y \cap U_p = f^{-1}(0)$ and $(df)_p \neq 0$.

Then $\exists$ local complex coordinates $z_1, \dots, z_n$ around p s.t. $f(z) = z_1$ (extend and use Inverse function thm)
$\therefore \forall$ holo $g : U_p \to \mathbb{C}$ s.t. $g|_{Y \cap U_p} \equiv 0$, then $g = fu$ (i.e. $f \mid g$)

- consider power series at p:

  g is a power series in $z_1, \dots, z_n$. But the function vanishes whenever $z_1 = 0$. So any terms in power series will have a z_1 factor. This can be shown by inducting on dimension

Thus we have an irreducible element $f \in \mathcal{O}_{X,p}$ s.t $\forall g$ vanishing on Y near p, we have $f \mid g$. Such an f is unique up to an invertible element. (†)

Definition: a subvariety $Y \subset X$ is (locally) irreducible at $p \in Y$ if $\exists$ small polydisc around p s.t $Y \cap U$ is irreducible.

Suppose $p \in Y^s$, and Y is irreducible at P (Y is a hypersurface).

Claim: $\exists$ $U_p \subset X$ open and $f: U_p \to \mathbb{C}$ holo such that $Y \cap U_p = f^{-1}(0)$.

wanna ask vanessa about this

Suppose the claim is false. We know $Y \cap U_p \subsetneq \left\{ q : \begin{matrix} f_1(q) = 0 \\ f_2(q) = 0 \end{matrix} \; ; \; f_j : U_p \to \mathbb{C} \text{ holo} \right\}$. Then we can assume f_1, f_2 are irreducible at p, so they are coprime at $p \Rightarrow$ coprime at any q near p by prop, in particular at some $q \in Y^*$, since Y^* is dense and open in Y. Since q is smooth, $f_1 = f_0 u$ and $f_2 = f_0 u$ in $\mathcal{O}_{X,q}$ for some $f_0 \in \mathcal{O}_{X,q}$ irreducible by Statement (†). This is a contradiction, since then f_1 and f_2 are not coprime at q.

□

If Y is not irreducible, then apply claim to each component and multiply out each of the functions

We obtain the following:

Definition / Proposition: Let $Y \subset X$ be a hypersurface, and $p \in Y$. Then $\exists$ $f \in \mathcal{O}_{X,p}$ such that $f|_{Y \cap U_p} \equiv 0$ and this f is unique up to invertible factors: for any other such g, $f \mid g$. We call f a local defining function for Y at p.

For $p \notin Y$, formally set f to be any invertible element at p.

Lemma: a hypersurface Y is irreducible at $p \Leftrightarrow$ local defining function at p is irreducible in $\mathcal{O}_{X,p}$.

proof: ($\Leftarrow$) Suppose f is irreducible, and for contradiction's sake Y is not irred. at p. If $Y \cap U_p = Y_1 \cup Y_2$ nontrivial, then $\exists$ local defining functions f_j for Y_j at p. By Nullstellensatz, $\Rightarrow f$ (vanishes on $Y \cap U_p$) must have $f \mid (f_1 f_2)$. But f is irreducible, so $f \mid f_1$ or $f \mid f_2$ by UFD. Suppose $f \mid f_1$. Then $Y_1 \supseteq Y$ or $Y_2 \supseteq Y$, which contradicts the fact that $Y_j \neq Y \cap U_p$. ↯

($\Rightarrow$) if f is a local def. function for Y, and $f = f_1 f_2$, f_1, f_2 coprime, then $Y \cap U_p = \{f_1 = 0\} \cup \{f_2 = 0\} =: Y_1 \cap Y_2$, where $Y_j \neq Y \cap U_p$.

□

In general, using local defining functions and UFD, we obtain $\forall p \in X$, $\exists$ open nhood $U_p \subset X$ s.t $Y \cap U_p = Y_{p,1} \cup \dots Y_{p,m}$ with each $Y_{p,j}$ irreducible.

If X is compact, can pass to a finite cover of X by U_p's, and can patch these $Y_{p,j}$'s and obtain a global decomposition

$$Y = Y_1 \cup \dots \cup Y_N \quad (*)$$

Where ach Y_j is a globally irreducible analytic hypersurface.

Definition: A divisor on a complex manifold X is a locally finite formal sum $\underset{\substack{\nearrow \\ \text{divisor}}}{D} = \sum_i a_i Y_i$ where each Y_i is an irreducible hypersurface in X, and $a_i \in \mathbb{Z}$.
locally finite means $\forall p \in X$, $\exists$ open nhood U_p such that D meets U_p in only finitely many Y_i's.

The set of divisors $\mathrm{Div}(X)$ forms a group under addition.

Let X be compact. Then $\exists$ finite open cover by U_p's, say $\{U_\alpha\}$, and for any α $\exists$ well-defined holomorphic local defining functions $f_{j,\alpha} : U_\alpha \to \mathbb{C}$ for $\forall j$ (recall $(*)$). Then we can assign to any divisor $D \in \mathrm{Div}(X)$ the data $\{(U_\alpha, f_\alpha)\}$ where $f_\alpha := \prod_{i=1}^{N} f_{\alpha,i}^{\,a_i} : U_\alpha \to \mathbb{C}$, called the "locally defining function" for D at $p \in U_\alpha$.

We define $D \in \mathrm{Div}(X)$ to be effective when $a_i \geq 0$ $\forall i$. Then f_α is holomorphic on U_α.

Definition: f is called a meromorphic function on X if locally, f is a quotient of two holomorphic functions. If $X = \bigcup_i U_i$, U_i open, then $\exists$ coprime holo $g_i, h_i : U_i \to \mathbb{C}$ with $h_i \not\equiv 0$ such that $f|_{U_i} = \frac{g_i}{h_i}$, and $g_i h_j = g_j h_i$ on any intersection $U_i \cap U_j$.

Basic example of a meromorphic function (in $\dim_{\mathbb{C}} > 1$).

$X = \mathbb{C}^n$, $g(z) = z_\alpha$, $h(z) = z_\beta$, $\alpha \neq \beta$, then $\frac{g(z)}{h(z)}$ is undefined on $\{z_\alpha = z_\beta = 0\}$, a $\mathrm{codim}_{\mathbb{C}} = 2$ subspace.
(unlike $\dim_{\mathbb{C}} = 1$ case, where meromorphic functions only have poles, so $\Leftrightarrow$ holo maps to $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$).

Let $Y \subset X$ be an irreducible analytic hypersurface, $p \in Y$, and f a local defining function at p. If g is a holo function around p, $g \not\equiv 0$, $g = g_1, \dots, g_l$ irreducible factors ($\exists$ by UFD), then define

Definition: $\mathrm{ord}_{Y,p}(g) :=$ $\max\{a \in \mathbb{Z} : g = f^a h \text{ for some } h \text{ holo at } p\}$
i.e. $f^a \mid g$ in $\mathcal{O}_{\mathbb{C}^n,p}$.

This concept is independent of f since f is unique up to invertible factors. and so is g.

Recall $Y^* = Y \setminus Y^{\overset{\text{singular locus}}{c}}$ is connected, open and dense in Y.

Claim: $\mathrm{ord}_{Y,p}(g)$ is locally constant for $p \in Y^*$, and thus independent of $p \in Y^*$. Hence $\mathrm{ord}_{Y,p}(g)$ is well-defined independent of p, and so we can drop p from the notation and write $\mathrm{ord}_Y(g)$.

proof: wlog $p = 0 \in \mathbb{C}^n = \overset{\text{coordinates}}{\mathbb{C}_w \times \mathbb{C}_z^{n-1}}$, and assume X is a polydisc around 0, and $Y = \{w = 0\}$. Then
$\mathrm{ord}_{Y,0}(g) = a \Leftrightarrow g(w,z) = w^a h(w,z)$, $w \nmid h$ in $\mathcal{O}_{\mathbb{C}^n,0}$
$\Rightarrow h = w h_0 + h_1$, where h_0, h_1 are holo near 0 and $\frac{\partial h_1}{\partial w} = 0$, and $h_1(0,0) \not\equiv 0$ represented by nontrivial power series $h(0,z) \not\equiv 0$ for small $|z|$. Re-expand h at $(0,z)$, still nontriv. power series.

Hence $\Rightarrow w \nmid h$ at $(0,z)$ if $|z|$ is small. $\square$

It is easy to see that $\mathrm{ord}_Y(gh) = \mathrm{ord}_Y(g) + \mathrm{ord}_Y(h)$ by UFD and noting f is irreducible.

If $F \not\equiv 0$ is meromorphic, then locally $F = \frac{g}{h}$, g, h holo.

Definition: $\mathrm{ord}_Y(F) := \mathrm{ord}_Y(g) - \mathrm{ord}_Y(h)$

if $d = \mathrm{ord}_Y F > 0$, then called a zero order d along Y
if $d = \mathrm{ord}_Y F < 0$, then called a pole order $(-d)$ along Y.

The divisor of a meromorphic function $F \not\equiv 0$ on X is

$$(F) := \sum_{\substack{Y \text{ irred} \\ \text{hypersurf} \\ \text{in } X}} \operatorname{ord}_Y(F) \cdot Y \quad , \qquad (*)$$

which is well-defined by U.F.D. Any such divisor is called a principal divisor.

Remarks:

- $D \sim D'$ "linearly equivalent" iff $D - D' = (F)$ for some meromorphic F.
- The sum $(*)$ is finite when X is compact
- $(F) \geqslant 0$ (effective) iff F is holomorphic
- $(FG) = (F) + (G)$, $\left(\frac{F}{G}\right) = (F) - (G)$, assume $g \not\equiv 0$

N.B. if $\dim_{\mathbb{C}} X = 1$, X compact , i.e. a Riemann surface , then $\operatorname{Div}(X) = \left\{ \sum_i n_i P_i \ , \ n_i \in \mathbb{Z}, \ P_i \in X \right\}$.
When $\dim_{\mathbb{C}} X > 1$, there need not be any divisors on X in general. But divisors always exist when X is projective.

Suppose $F : Z \to X$ is a holo map of manifolds. Assume Z and X are compact and connected. Let $D \in \operatorname{Div}(X)$, $D = \sum_i a_i Y_i$, assume $f(Z) \not\subset Y_i \ \forall i$ with $a_i \neq 0$. Then $F^* D \in \operatorname{Div}(Z)$ is well defined.

Recall $X = \bigcup_\alpha U_\alpha \quad \forall i \ \forall \alpha$, $f_{\alpha i} : U_i \to \mathbb{C}$ where $f_{\alpha i}$ is a local defining function for Y_i.
The "data" of $D \Leftrightarrow \{(U_\alpha, f_\alpha)\}$; $f_\alpha = \prod_i (f_{\alpha i})^{a_i}$ for D is meromorphic.
"Cartier divisor"

Then $F^* D$ corresponds to $\left\{ \left(F^{-1}(U_\alpha), \ f_\alpha \circ F \right) \right\}$

N.B when $D = Y$ is an irreducible hypersurface in X, the $F^* D$ need not be irreducible and may have "multiplicities"

Given D; $\{(U_\alpha, f_\alpha)\}$ define $\varphi_{\beta\alpha} = \frac{f_\beta}{f_\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}$, quotient is an invertible factor and so $\varphi_{\beta\alpha}$ is holo , and moreover nonzero on $U_\alpha \cap U_\beta$
Clearly $\varphi_{\alpha\beta} \varphi_{\beta\gamma} \varphi_{\gamma\alpha} \equiv 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$, an instance of the cocycle condition.
$\Rightarrow$ determines a holomorphic line bundle over X, denoted by $[D] \in \operatorname{Pic}(X)$, called an associated line bundle to D, $D \in \operatorname{Div}(X)$.

Remark : $[D]$ is well defined : ambiguity $\tilde{f}_\alpha = h_\alpha f_\alpha$ for some invertible h holo (never zero). So
$\tilde{\varphi}_{\beta\alpha} = \varphi_{\beta\alpha} \frac{h_\beta}{h_\alpha}$,$[\tilde{D}] \cong [D] \otimes L \cong [D]$, Since L has a never zero holo section $\Rightarrow$ L is trivial

Indeed we define $h : X \to L$, holo never zero, $h|_{U_\alpha} = h_\alpha : U_\alpha \to \mathbb{C}$, $h_\beta = \varphi_{\beta\alpha} h_\alpha : U_\beta \to \mathbb{C}$,

- If $D = (f)$, then $f_\alpha = f|_{U_\alpha} = \prod \frac{g_{\alpha i}}{h_{\alpha i}}$, ? $\varphi_{\beta\alpha} \equiv 1$ thus $[D]$ is holomorphically trivial
- If $[D]$ is holomorphically trivial, then $[D]$ has a neverzero holo section s. Then over U_α, s is represented by $s_\alpha : U_\alpha \to \mathbb{C} \setminus \{0\}$, with $s_\beta = \varphi_{\beta\alpha} s_\alpha$ on $U_\alpha \cap U_\beta$, so $\frac{s_\alpha}{s_\beta} = \varphi_{\alpha\beta} \overset{\text{dfn}}{=} \frac{f_\alpha}{f_\beta}$ by dfn of $[D]$.

Consider then that $\frac{f_\alpha}{s_\alpha} = \frac{f_\beta}{s_\beta}$, which patch together to give a well defined (global) mero function f on all of X, and hence $D = (f)$

I.e. D is a principal divisor iff it's associated line bundle $[D]$ is trivial. Also, $[D] = [\tilde{D}]$ in $\operatorname{Pic}(X)$ iff $D \sim \tilde{D}$ (linearly equivalent $\sim :=$ difference is a principal divisor)

- Consider $D+\tilde{D}$ has local defining functions $f_\alpha \cdot \tilde{f}_\alpha$ $\left(\begin{matrix} f_\alpha & \text{for } D \\ \tilde{f}_\alpha & \text{for } \tilde{D} \end{matrix}\right)$. Hence $[D+\tilde{D}] = [D] \otimes [\tilde{D}]$

We have thus proved:

Proposition: $D \in \mathrm{Div}(X) \to [D] \in \mathrm{Pic}(X)$ is a group homomorphism. The kernel is the principal divisors. (†)

- if $F: Z \to X$ is a holo map of manifolds and $F^*D \in \mathrm{Div}(Z)$ is well defined, then $F^*[D] = [F^*D]$ by considering (pullbacks of) the local defining functions and how they give rise to transition functions.

Recall: a section of holo line bundle $L \to X$ is holo if it is expressed by holo functions in each holo. local trivialisation.

We can similarly define meromorphic sections of L (i.e. loc. expressed as a meromorphic function in each holo local trivialisation).

Basic properties:

- if $s_0 \not\equiv 0$ and s are two meromorphic sections of L $\Rightarrow$ $s = f s_0$ for some mero function f
- conversely, if f is a mero section and f mero function on X, then fs is a mero section.

Hence by choosing $s_0 \not\equiv 0$, obtain a linear isomorphism $f \mapsto f s_0$ between

$$\{\text{mero sections on } U_0 \subseteq X\} \leftrightarrow \{\text{mero functions on } U_0\}$$

- let $s \not\equiv 0$ a mero section. $s_\alpha := s|_{U_\alpha}$, where U_α is a trivialising nbhood for L. Then $\frac{s_\alpha}{s_\beta} = \psi_{\alpha\beta}$, which is holo and never zero function

$\Rightarrow$ $\forall$ irreducible hypersurfaces $Y \subset X$, then $\mathrm{ord}_Y(s_\alpha) = \mathrm{ord}_Y(s_\beta)$ on $U_\alpha \cap U_\beta$. Hence globally, $\mathrm{ord}_Y(s)$ is well-defined.

Therefore $(s) \in \mathrm{Div}(X)$ is well defined as $(s) := \sum_{\text{irred } Y} \mathrm{ord}_Y s \cdot Y$, which generalizes divisors to meromorphic functions.

- $(s) \geq 0$ means s is a holo section

- D corresponds to $\overset{\text{Cartier divisor}}{\{(U_\alpha, f_\alpha)\}}$, $f_\beta = \psi_{\beta\alpha} f_\alpha$ by definition of $[D]$. We can then infer that $\exists$ a mero section of $[D]$ given by imposing $s|_{U_\alpha} = f_\alpha$, and so $(s) = D$ (seen by reversing above argument). In particular, $[(s)] = [D]$ in $\mathrm{Pic}(X)$.

Furthermore we obtain $\forall$ mero sections s of L, $L = [(s)]$. Conversely, given $L \to X$ a holo line bundle

(1) $\{D \in \mathrm{Div}(X) : [D] = L\} \cong \{\text{nonzero mero sections of } L\} / \mathbb{C}^*$

(up to holo isomorphism) (multiplying by nonzero constant gives same divisor)

The image of map in Proposition (†) $D \in \mathrm{Div}\, X \to [D] \in \mathrm{Pic}(X)$ is the subgroup of $\mathrm{Pic}(X)$ of line bundles admitting nontrivial mero sections.

(2) $\mathcal{L}(D) = \{\text{mero functions on } X : D + (f) \geq 0\} \cup \{0\}$ is a vector space $\cong \{\text{v.s. of all holo sections of } [D]\}$.

Fact: $\dim(\mathcal{L}(D)) < \infty$ when X is compact.

Remark: $\exists$ complex manifold X ($\dim X \geq 2$) without divisors but still with holo bundles.

The First Chern Class

Let $L \to X$ be any smooth complex line bundle over complex manifold (assume X is compact). Let d_A be a covariant derivative (corresponding to a connection A). Recall that $d_A : \Gamma(L) \longrightarrow \Gamma(T^*X \otimes L) =: \Omega^1_X(L)$. More generally, $d_A : \Omega^r_X(L) \to \Omega^{r+1}_X(L)$. Locally, $d_A s_\alpha = d s_\alpha + A_\alpha s_\alpha$ in a trivialisation over $U_\alpha \subset X$, with A_α are the local differential forms expressing A, $A_\alpha \in \Omega^1(U_\alpha)$ $(s_\alpha = s|_{U_\alpha})$.

There is a transformation law: $A_\beta = A_\alpha + \psi_{\beta\alpha}\, d\psi^{-1}_{\beta\alpha}$ on $U_\alpha \cap U_\beta$ for $\psi_{\beta\alpha}$ smooth. $(*)$

Recall curvature $d_A d_A\, s = F(A) s$, where $F(A) \in \Omega^2_X(\operatorname{End}(L)) \underset{\nearrow}{\cong} \Omega^2(X)$

L rank 1 so $\operatorname{End}(L)$ is mult by Complex #

So locally $F(A)|_{U_\alpha} = dA_\alpha$ $([A,A]=0$ in this case$)$

Then $dF(A) = 0$ (closed is local condition). Curvature is exact locally (true by Poincaré lemma), but not exact globally necessarily due to $(*)$ (need not be exact)

Any other connection on L is $A + a$ where $a \in \Omega^1(X)$. So $F(A+a) = F(A) + da$. Thus $[F(A)] \in H^2(X;\mathbb{C}) \cong H^2_{dR}(X) \otimes \mathbb{C}$ is well defined independent of auxillary choice of A. It depends only on L.

We can choose a Hermitian inner product $\langle\cdot,\cdot\rangle$ on the fibres of L. Suppose a connection A is unitary :

$$d\langle s_1, s_2\rangle = \langle d_A s_1, s_2\rangle + \langle s_1, d_A s_2\rangle \qquad \forall\, s_1, s_2 \in \Gamma(L)$$

Then in a unitary local trivialisation, A_α are skew-Hermitian. $\Rightarrow$ A_α is a scalar $(\operatorname{rk} L = 1)$ so A_α is pure imaginary.
(inner product locally is represented by identity matrix)

Hence $\frac{iF(A)}{2\pi}$ is a real form, so $\left[\frac{iF(A)}{2\pi}\right] \in H^2_{dR}(X)$, denoted $c_1(L)$, defines the 1st Chern class of L.

Proposition: $c_1(L \otimes \tilde{L}) = c_1(L) + c_1(\tilde{L})$
in particular, $c_1(L^\vee) = -c(L)$ ($\vee$ and $*$ denotes dual)

proof: Consider sections $s \in \Gamma(L)$, $\tilde{s} \in \Gamma(\tilde{L})$. Then $s \otimes \tilde{s}$ is represented over each triv. nbhood U_α (for both L and $\tilde{L}$) by $s_\alpha \cdot \tilde{s}_\alpha$ (locally). Since transitions are also multiplied for the tensor bundle, $s_1 \otimes \tilde{s} \in \Gamma(L \otimes \tilde{L})$ is well-defined.

Let $A, \tilde{A}$ be connections respectively on $L, \tilde{L}$. We can define $d_{A\otimes\tilde{A}}(s\otimes\tilde{s}) \underset{\text{locally}}{:=} (ds + As)\otimes\tilde{s} + s\otimes(d\tilde{s} + \tilde{A}\tilde{s})$
(defining $A \otimes \tilde{A}$ by it's covariant derivative) $\qquad d_A s \otimes \tilde{s} + s \otimes d_{\hat{A}} \tilde{s}$

In a trivialisation, $= d(s_\alpha \tilde{s}_\alpha) + (A_\alpha + \tilde{A}_\alpha)\cdot s_\alpha \tilde{s}_\alpha$

Then $d_{A\otimes\hat{A}}\left(d_{A\otimes\tilde{A}}(s\otimes\tilde{s})\right) = (F(A) + F(\tilde{A}))\cdot(s\otimes\tilde{s})$
locally $= d(A_\alpha + \tilde{A}_\alpha)\cdot s_\alpha \tilde{s}_\alpha$.

Thus on $L \otimes \tilde{L}$ we obtain $\left[\frac{i}{2\pi}(F(A) + \tilde{F}(A))\right]$ I.e. $c_1(L\otimes\tilde{L}) = c_1(L) + c_1(\tilde{L})$.

For the last part, note that the trivial line bundle $L^\vee \otimes L$ admits a global trivialisation, so has a 1-form representing A defined on all of X. Therefore $F(A)$ is exact $\Rightarrow$ dR class is trivial. Combining this with the previous result, we get $c_1(L^\vee) = -c_1(L)$ $\square$

Proposition (Chern connection for special case of line bundles)

Suppose L is a holo line bundle, with Hermitian inner product on fibres. Then $\exists!$ connection A on L such that

(i) A is unitary

(ii) in any holo trivialisation of L over say $U_\alpha \subseteq X$, $A_\alpha \in \Omega^{1,0}(U_\alpha)$.

Proof: wlog U_α is also a coordinate nbhd. Consider a local holo section $e : U_\alpha \to \mathbb{C}$, $e(z) \equiv 1$, where z is the coordinates, $z \in \mathbb{C}^n$. The hermitian product $h_\alpha(z) = |e(z)|^2 = \langle e(z), e(z) \rangle$ (we'll drop the α from $h_\alpha, U_\alpha, A_\alpha$ for ease of notation). Then any section over U is λe for $\lambda : U \to \mathbb{C}$.

(i) we require

$$\begin{aligned} d|s|^2 &= \langle d_A s, s \rangle + \langle s, d_A s \rangle \\ &= \langle (d\lambda + A\lambda)e, \lambda e \rangle + \langle \lambda e, (d\lambda + A\lambda)e \rangle. \qquad \text{(over each } U\text{)} \\ &= h\bar{\lambda}\, d\lambda + h\lambda\, d\bar{\lambda} + h|\lambda|^2(A + \bar{A}). \end{aligned}$$

Also have $d|s|^2 = h\lambda\, d\bar{\lambda} + h\bar{\lambda}\, d\lambda + |\lambda|^2\, dh$ by product rule for exterior derivative.

Hence we must have that $A + \bar{A} = \frac{dh}{h}$. h is an inner product so nonvanishing function.

(ii) requires A $(1,0)$-form and $\bar{A}$ a $(0,1)$-form.

Then for both of these, we need $A^{1,0} = \frac{\partial h}{h} = \partial \log h$ on U.

For any other holo local trivialisation over U_β say, s.t. $U_\beta \cap U \neq \phi$, $\bar{\partial}\Psi_{\alpha\beta} = 0$ (holo). So $d\Psi_{\alpha\beta} = \partial\Psi_{\alpha\beta}$, and $\partial\overline{\Psi_{\alpha\beta}} = 0$

Then $h_\beta = \Psi_{\alpha\beta}\overline{\Psi_{\alpha\beta}}\, h$ (how hermitian product from linear algebra changes under change of basis matrix. But since these are 1×1 matrices (line bundle), the order of multiplication doesnt matter)

Thus

$$A_\beta = \partial \log h_\beta = \partial \log h + \partial\left(\frac{\Psi_{\beta\alpha}^{-1}\overline{\Psi_{\beta\alpha}^{-1}}}{\Psi_{\beta\alpha}^{-1}\overline{\Psi_{\beta\alpha}^{-1}}} \right)$$

$$= A + \Psi_{\beta\alpha}\, d\Psi_{\beta\alpha}^{-1}$$ which is (*) from last lecture.

$\square$

Corollary 1: The curvature of the Chern connection

$$F(A) = \bar{\partial}\partial \log|e|^2 = \frac{i}{2} dd^c \log|e|^2$$

where e is any local holo section of L without zeros.

note: e <u>not</u> global, but local. RHS local expression, but patch together.

pf: Exercise

Exercise: explain why $F(A)$ is not in general $\bar{\partial}$-exact.

Corollary 2: $\frac{i}{2\pi}[F(A)] = c_1(L) \in H^{1,1}(X)$

from Corollary 1

Remark: in topology, c_1 of complex v.b. is defined as a class in $H^1(X;\mathbb{Z})$. But $H_{dR}^*(X) \cong H^*(X;\mathbb{R})$ and $\mathbb{Z} \subset \mathbb{R}$ induces a homomorphism

22/02

Convention: X compact, connected complex n-fold.

Consider $Y \subset X$ analytic hypersurface ($\Rightarrow$ Y also compact)

Then $\forall \varphi \in \Omega^{2n-2}(X)$ with $d\varphi = 0$ (closed) then consider the linear functional

$$[\varphi] \in H_{dR}^{2n-2}(X) \longrightarrow \int_{Y^*} \varphi \in \mathbb{R}$$

use the orientation on Y^* from complex structure.

independent of representative by Stokes'.

By the Poincaré duality, $\exists !\ \eta_Y \in \Omega^2_{dR}(X)$ s.t $\int_{Y^*} \varphi = \int_X \eta_Y \wedge \varphi$

i.e. $\eta_Y = \text{P.D.}[Y]$, $[Y] \in H_{2n-2}(X,\mathbb{R})$, image of $[Y] \in H_{2n-2}(X,\mathbb{Z})$.

Then $\forall D \in \text{Div}(X)$, $D = \sum_i a_i Y_i$ (finite), define $\eta_D = \sum_i a_i \eta_{Y_i} \in H^2_{dR}(X)$

Proposition: $\eta_D = c_1([D])$.

Corollary: $c_1([D])$ is in the image of the natural homomorphism $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R}) \cong H^2_{dR}(X)$.

proof of proposition: to show: $\forall \varphi \in \Omega^{2n-2}(X)$ s.t $d\varphi = 0$, we have the following:

$$\frac{i}{2\pi} \int_X F(A) \wedge \varphi = \sum_i a_i \int_{Y_i} \varphi$$

Where $\sum a_i Y_i = D$, and A is a connection on $[D]$. Let A be the Chern connection for some norm $|\cdot|$ on the fibres of $[D]$.

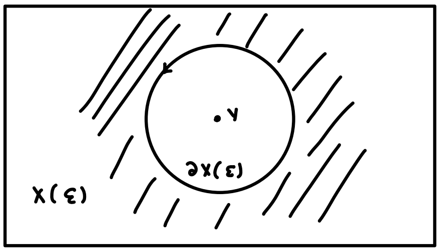
Wlog (linearity) take $D = Y$ ("one" hypersurface) wlog Y not singular otherwise use smooth locus.

Let $X = \bigcup_{\alpha=1}^{N} U_\alpha$, f_α local defining function for our hypersurface Y on U_α. I.e. $Y = (s)$, where s is a meromorphic section of $[D] = [Y]$, and $f_\alpha = s_\alpha$ ($= s|_{U_\alpha}$ in local trivialisation).

Since our divisor is effective, in fact s is a holo section.

Put $X(\varepsilon) = \{ p \in X : |s(p)| > \varepsilon \}$, $\varepsilon > 0$ (complement of tubular nhood of Y in X of size ε)

Diagram: cross-section transverse to hypersurface



Then $\int_X F(A) \wedge \varphi = \frac{i}{2} \lim_{\varepsilon \to 0} \int_{X(\varepsilon)} (dd^c \log|s|^2) \wedge \varphi$ (*)

By Stoke's thm exact ∧ closed = exact, we can then integrate over boundary

$$(*) = -\frac{i}{2} \lim_{\varepsilon \to 0} \int_{\partial X(\varepsilon)} (d^c \log|s|^2) \wedge \varphi.$$

Look at integrand: $|s|^2 \big|_{U_\alpha \cap (X \setminus X(\varepsilon))} = |f_\alpha|^2 h_\alpha = f_\alpha \bar{f_\alpha} h_\alpha$. go back in recording

For some $h_\alpha > 0$ the local expression for Hermitian norm on the fibres of $[D]$ on U_α.

Hence on $U_\alpha \cap (X \setminus X(\varepsilon))$, $d^c \log|s|^2 = i(\bar\partial - \partial) \log(f_\alpha \bar f_\alpha h_\alpha)$

$= i\,(\bar\partial \log \bar f_\alpha - \partial \log(f_\alpha) + (\bar\partial - \partial)\log(h_\alpha))$ $\quad f_\alpha$ holo

Notice that $\mathrm{vol}(\partial X(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Also h_α is bounded away from 0, and similarly ∇h_α is also bounded on $\bar U_\alpha$. Hence

$$\lim_{\varepsilon \to 0} \int_{\partial X(\varepsilon) \cap U_\alpha} (d^c \log h_\alpha) \wedge \varphi = 0$$

extends to whole of $\partial X(\varepsilon)$.

Since φ is a real differential form, we can write

$$\int_{\partial X(\varepsilon) \cap U_\alpha} (\bar\partial \log \bar f_\alpha) \wedge \varphi = \overline{\int_{\partial X(\varepsilon) \cap U_\alpha} (\partial \log f_\alpha) \wedge \varphi}$$

Therefore taking the limit

$$(\dagger) \quad \lim_{\varepsilon \to 0} -\frac{i}{2} \int_{\partial X(\varepsilon) \cap U_\alpha} (d^c \log|s|^2) \wedge \varphi = \lim_{\varepsilon \to 0} -i \,\mathrm{Im}\left(\int_{\partial X(\varepsilon) \cap U_\alpha} (d \log f_\alpha) \wedge \varphi \right)$$

Choose local coordinates on U_α such that $f_\alpha(z) = z_1$ $\quad (z = (z_1, \dots, z_n))$. (Can assume $U_\alpha \cong$ a coord. polydisc in $\mathbb{C}^n$).

Can then decompose $\varphi = \tilde\varphi + \varphi_1$, $\tilde\varphi$ is all of the summands containing dz_1 or $d\bar z_1$. Then

$$(\dagger) = -i \,\mathrm{Im}\left(\lim_{\varepsilon \to 0} \int_{|z_1| = \frac{\varepsilon}{\sqrt{h_\alpha^2}}} \frac{dz_1}{z_1} \wedge \varphi_1(z_1, \dots) \right)$$

$$= -i \; 2\pi \left(\int_{|z_1| = 0} \varphi_1(0, z_2, \dots) \right) \quad \text{i.e. the residue.}$$

Thus, patch over U_α's and sum up,

$$\lim_{\varepsilon \to 0} \frac{i}{2} \int_{\partial X(\varepsilon)} (d^c \log|s|^2) \wedge \varphi = -2\pi i \int_Y \varphi$$

I.e. $\int_X F(A) \wedge \varphi = -2\pi i \int_Y \varphi$

$\square$

Examples:

1) $X = S$ Compact connected Riemann surface ($\dim_{\mathbb{C}} X = 1$).
Then $D = \sum a_i P_i$ for points $P_i \in S$, $D \in \mathrm{Div}(S)$.
$\forall\, P \in S$ is a generator $[P]$ of $H_0(S;\mathbb{Z})$, inducing a group homomorphism
$\mathrm{Div}\, S \to \mathbb{Z}$ given by degree map, i.e. $\deg D = \sum a_i$.

Then $\forall\, p \in S$, $\eta_p : P.D[P] \in H^2(S,\mathbb{Z}) \cong \mathbb{Z}$. generated by $\eta_p = [\varphi]$, $\varphi \in \Omega^2(S)$, $\int_S \varphi = 1$.

If L is a complex line bundle over S, define $\deg L := \langle c_2(L), [S] \rangle \in \mathbb{Z}$, $[S] \in H_2(S,\mathbb{Z})$
fundamental class.

So if $L = [D]$, then $\deg [D] = \frac{-1}{2\pi i} \int_S F(A) = \deg D$ by proposition.

Remark : as $\deg : \mathrm{Div}(D) \to \mathbb{Z}$ is clearly surjective, $\exists$ holo line bundles with mero sections over S for each value of c_1 as $H^2(S,\mathbb{Z}) \cong \mathbb{Z}$.

Recall: for $L \in \mathrm{Pic}(S)$, S a complex manifold of $\dim_{\mathbb{C}} = 1$, compact. Then $\deg(L)$ is defined as

$$\deg(L) = \langle c_1(L), [S] \rangle \in \mathbb{Z}$$, where $[S]$ = fundamental cycle $\in H_2(S,\mathbb{Z})$.
(called 1^{st} Chern number in topology)

Our Proposition asserts then that if $L = [D]$, then $\underset{[D] \in \mathrm{Pic}(S)}{\deg([D])} = \frac{-1}{2\pi i} \int_S F(A) = \underset{D \in \mathrm{Div}(S)}{\deg D}$.

Then $\deg : \mathrm{Div}(S) \to \mathbb{Z}$ is obviously a surjective homomorphism. Hence $\exists$ holomorphic line bundles with $\not\equiv 0$ mero sections over S with any $c_1(S)$ (i.e. any degree).

Now let $S = \mathbb{CP}^1$ (Riemann sphere) $\cong \mathbb{C} \cup \{\infty\}$.

- Nonconstant holo maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ are precisely the rational functions
- Every rational function has the same # zeros and # poles in $\mathbb{C} \cup \{\infty\}$ (counting with multiplicities)
- $\sum_{\text{finite}} a_i P_i \sim \sum_{\text{finite}} b_j Q_j$ (linear equiv in $\mathrm{Div}(\mathbb{CP}^1)$) $\Leftrightarrow \sum a_i = \sum b_j$

Consider $\mathbb{C}^2 \setminus \{(0,0)\} \xrightarrow{\pi} \mathbb{CP}^1$, the Hopf bundle $\mathcal{O}(-1)$. Then the map $[z_1 : z_2] \to (1, z_1/z_2)$ from $\mathbb{CP}^1 \to \mathbb{C}^2 \setminus \{(0,0)\}$ induces a mero section of $\mathcal{O}(-1)$. Locally $s_1 \equiv 1$ on $U_1 = \{ z_1 \neq 0 \} \subset \mathbb{CP}^1$, and $s_2([z:1]) = 1/z$ on $U_2 = \{ z_2 \neq 0 \}$. Thus $\exists!$ pole at $0{:}1$ and has order 1. Thus divisor is $D = (-1)[0:1]$, and $\deg(\mathcal{O}(-1)) = -1$. In general, $\deg \mathcal{O}(K) = K$.

Proposition: Let $L \to \mathbb{CP}^1$ be a holo line bundle, and assume $c_1(L) = 0$. Then L is holomorphically trivial.

Corollary: $\mathrm{Pic}(\mathbb{CP}^1) = \{ \mathcal{O}(n) : n \in \mathbb{Z} \} \cong \mathbb{Z}$.

proof of proposition:
If $c_1(L) = 0$, $\Rightarrow L$ is trivial as a smooth bundle since $F(A)$ is exact and A can be represented by a 1-form making sense globally over $\mathbb{CP}^1$, thus get a global trivialisation.

Fix a nonvanishing smooth section s of L. Then $L \overset{\text{in smooth sense}}{\xrightarrow{\cong}} \mathbb{CP}^1 \times \mathbb{C}$. Choose a Hermitian norm on fibres and let A be the Chern connection.

Then $d_A = \underset{\substack{(1,0)\\ \text{component}}}{\partial_A} + \underset{\substack{(0,1)\\ \text{component.}}}{\bar\partial_A}$

Then s is holo iff $\bar\partial_A s = 0$. We want a global, never zero section s of L with $\bar\partial_A s = 0$.

Consider $s = e^f : \mathbb{CP}^1 \to \mathbb{C}$, f smooth. Then $\bar\partial_A s = \bar\partial s + A'' s = 0 \iff \bar\partial f = -A''$

$A'' \in \Omega^{0,1}(\mathbb{CP}^1)$

Consider $\mathbb{CP}^1 = U_1 \cup U_2$ as before, where we think of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, $U_1 = \mathbb{C}$, and $U_2 \cdot \mathbb{C}^* \cup \{\infty\}$. Coordinate z on U_1 and $\zeta = 1/z$ on U_2.

$\exists$ local solution $f_j : U_j \to \mathbb{C}$ with $\bar\partial f = -A''|_{U_j}$, $j = 1,2$ by $\bar\partial$-Poincaré lemma. Hence $\bar\partial(f_1 - f_2) = 0$ on $\mathbb{C}^*$. So write $f_1(z) - f_2(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ $\forall z \neq 0$. Then let $f = \begin{cases} f_1 + \sum_{n=0}^{\infty} c_n z^n & \text{on } U_2 \\ f_2 + \sum_{n=-\infty}^{0} c_n z^n & \text{on } U_1 \end{cases}$

Well defined on $\mathbb{CP}^1$ globally, and solves $\bar\partial f = -A''$.

$\square$

Remarks :
- in fact $\text{Pic}(\mathbb{CP}^n) = \{\mathcal{O}(k)\} \cong \mathbb{Z}$ $\forall n \geq 1$.
- in particular, $\mathcal{O}(1)$ has holo section s with $(s) = H_0$ for a hyperplane $H_0 = \{\underline{z} \in \mathbb{CP}^n : z_0 = 0\} \cong \mathbb{CP}^{n-1}$.

  Identify $H^2(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}$, then $c_1(\mathcal{O}(k)) = k$ via this isomorphism.
- (EX 3, q 9) Compute Pic for $E = \mathbb{C}/\Lambda$, $\text{Pic}(E) \cong \mathbb{Z} \oplus E$ (E as additive group)

  In general it's not true that a topologically trivial line bundle is also a holomorphically trivial line bundle, even in $\dim_{\mathbb{C}} = 1$.

Definition: Consider a nonsingular, smooth, analytic hypersurface $Y \subset X$. The normal bundle $N_{Y/X} = \frac{T^{1,0}(X)|_Y}{T^{1,0}Y}$ the quotient bundle. The fibre of this bundle is $T_p^{1,0}X / T_p^{1,0}Y$ $\forall p \in Y$.

The conormal bundle $N^*_{Y/X}$ is the dual of $N_{Y/X}$. The fibre at $p \in Y$ is $\{\alpha \in (T_p^{1,0}X)^* : \alpha|_{T_p^{1,0}Y} = 0\}$
The conormal bundle is then a subbundle of $(T^*X)^{1,0}|_Y$. (exercise to show this).

Consider f_α local defining functions on $U_\alpha \subset X$. Consider $df_\alpha|_{Y \cap U_\alpha} = 0$ since f vanishes but $(df_\alpha)_p$ does not vanish $\neq 0$ in $(T_pX)^{1,0}$ $\forall p \in Y \cap U_\alpha$ (since nonsingular Y). Hence df_α defines a local never zero holo section of $N^*_{Y/X}$.

Recall can think of Y as a divisor, which gives rise to a line bundle with transition functions $\varphi_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ of $[Y]$.
The transition functions for $N^*_{Y/X}$ are $\tilde\varphi_{\alpha\beta} = \varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1}$. To see this,
as $df_\alpha = d(\varphi_{\alpha\beta} f_\beta) = d(\varphi_{\alpha\beta}) f_\beta + \varphi_{\alpha\beta} d(f_\beta)$. f_β vanishes on $Y \cap U_\alpha \cap U_\beta$, so

$= \varphi_{\alpha\beta} d(f_\beta)$

Thus $s_\beta df_\beta = s_\alpha df_\alpha$ iff $s_\alpha = \varphi_{\alpha\beta}^{-1} s_\beta$. and $[Y]|_Y \otimes N^*_{Y/X}$ is holomorphically trivial.

$\square$

Recall: have proved Adjunction formula I: $N_{Y/X}^* = ([Y]|_Y)^{-1} = [-Y]|_Y$

If f_α is a local defining function of Y on U_α, then f_α extends to local complex coordinates (wlog) on U_α $\{f_\alpha, \zeta_2, \dots \zeta_n\}$ (using Y is a nonsingular hypersurface). These $\zeta_2, \dots \zeta_n$ make sense as local coords on Y

Any holo local section of K_X on U_α is $h\, df_\alpha \wedge \omega_{Y,\alpha}$, where $\omega_{Y,\alpha}$ is an $(n-1)$-form, i.e. a local section of K_Y (pulled back onto X via the projection $(f,\zeta) \to \zeta$).

On $U_\alpha \cap U_\beta$, we have $\zeta^{(\beta)} = F_{\beta\alpha}(\zeta^{(\alpha)})$, and $f_\beta = G_{\beta\alpha}(f_\alpha, \zeta^{(\alpha)}) f_\alpha$ with $G_{\beta\alpha}(0, \zeta^{(\alpha)}) = \psi_{\beta\alpha}(\zeta^{(\alpha)})$ a transition function of the holo line bundle associated with Y as a divisor.

We find

$$K_X|_Y = N^*_{Y/X} \otimes K_Y.$$

This is the Adjunction formula II: $K_Y \cong (K_X \otimes [Y])|_Y$

We can use this to determine K_Y for hypersurfaces in $\mathbb{CP}^n$, more generally, many projective manifolds.

The canonical bundle of $\mathbb{CP}^n$

Coords: $[z_0 : \dots : z_n]$, with nhoods: $U_0 = \{z_0 \neq 0\} \subset \mathbb{CP}^n$, complex local coords: $w_i = \frac{z_i}{z_0}$, $i = 1, \dots, n$.
Can write a mero section of $K_{\mathbb{CP}^n}$ over U_0:

$$\omega = \frac{dw_1}{w_1} \wedge \dots \wedge \frac{dw_n}{w_n}$$

Define $\mathcal{H}_j = \{z_j = 0\} \subset \mathbb{CP}^n$. Corresponds to vanishing of w_j in our complex coords. Then clearly over every such $(j = 1, \dots, n)$ hyperplane, our mero section will have a pole of order 1. Hence $\operatorname{ord}_{\mathcal{H}_j} \omega = -1$ for $j = 1, \dots, n$.

Again, $U_j = \{z_j \neq 0\}$, can write local coords $\tilde{w}_k = \frac{z_k}{z_j}$, $k \neq j$. Then the relation between local coords is

$$w_i = \frac{\tilde{w}_i}{\tilde{w}_0} \quad \begin{pmatrix} i \neq j \\ i \neq 0 \end{pmatrix}$$

$$w_j = \frac{1}{\tilde{w}_0}$$

Then $\frac{dw_i}{w_i} = \frac{d\tilde{w}_i}{\tilde{w}_i} - \frac{d\tilde{w}_0}{\tilde{w}_0}$ with $i \neq j$, $i \neq 0$, and $\frac{dw_j}{w_j} = \frac{d\tilde{w}_0}{w_0}$. Substituting into formula for ω:

$$\omega = (-1)^j \frac{d\tilde{w}_0}{\tilde{w}_0} \wedge \dots \hat{j} \dots \wedge \frac{d\tilde{w}_n}{\tilde{w}_n}$$

Thus there are $n+1$ simple poles, one along each hyperplane $\mathcal{H}_i$ for $i = 0, \dots, n$. But since we know $\frac{z_i}{z_j}$ is a mero function on $\mathbb{CP}^n$ (associated divisor is principal, i.e. a unit in $\operatorname{Div}(\mathbb{CP}^n)$), then any hyperplanes $\mathcal{H}_i$ are linearly equivalent. $\mathcal{H}_i \sim \mathcal{H}_j$ in $\operatorname{Div} \mathbb{CP}^n$ $\forall\, i,j$. Hence we can think of this as one pole of order $n+1$. Therefore

$$K_{\mathbb{CP}^n} = [(\omega)] = [-(n+1)\mathcal{H}] = \mathcal{O}(-n-1).$$

Similar question in example sheet, using Q6 in Ex2. as $\mathcal{O}(-1) = [-\mathcal{H}_0]$, mero section is locally given by $s|_{U_0} \equiv 1$, $s|_{U_j} = \frac{z_j}{z_0}$. So ! simple pole.

Blow - Up

↗ informal dfn.

Consider a polydisc $\Delta \subset \mathbb{C}^n$ about 0. Write the blow up of Δ as

$$\tilde{\Delta} = \{ (z,w) \in \Delta \times \mathbb{CP}^{n-1} : z_i w_j = z_j w_i \quad \forall\, i,j \}$$

Claim: $\tilde{\Delta}$ is a complex manifold.

This condition $z_i w_j = z_j w_i$ says that z lies on the line defined by the point w of $\mathbb{CP}^{n-1}$: $\frac{z_i}{z_j} = \frac{w_i}{w_j}$.

Charts: for each standard chart $h_j : U_j \subset \mathbb{CP}^{n-1} \to \mathbb{C}^{n-1}$, put

$$\hat{h}_j : (z,w) \in (\Delta \times U_j) \cap \tilde{\Delta} \longrightarrow (h_j(w), z_j)$$

Straightforward check that these are well defined on overlaps.

Definition: $\sigma : \tilde{\Delta} \to \Delta$; $(z,w) \longmapsto z$ is called the blow up of Δ at 0.

Observe $\tilde{\Delta} \setminus \sigma^{-1}(0)$ is mapped biholomorphically onto $\Delta \setminus \{0\}$, and $\sigma^{-1}(0) \cong \mathbb{CP}^{n-1}$ easy check.
Informally, $\tilde{\Delta}$ means "lines through 0 in Δ are made distinct"

Remark :
- The blow up is trivial in $\dim_{\mathbb{C}}$, $n=1$.

- Let $\Delta = \mathbb{C}^n$. Then the second projection is a map $\tilde{\mathbb{C}}^n \to \mathbb{CP}^{n-1}$ realizes $\mathcal{O}(-1)$. The charts of $\tilde{\mathbb{C}}^n$ correspond to local trivialisations of $\mathcal{O}(-1)$.

- Can generalize to ll manifolds, say X: Consider $x \in X$, $U \subset X$ a coord polydisc with chart $\varphi : U \xrightarrow{\text{onto}} \Delta \subset \mathbb{C}^n$, with $x \in U$ and $\varphi(x) = 0$.

Put $\tilde{X} = (X \setminus \{x\}) \cup_{\varphi^{-1} \circ \sigma} \tilde{\Delta}$, identifying $\tilde{\Delta} \setminus \sigma^{-1}(0) \cong U \setminus \{x\}$. We obtain a holo map $\pi : \tilde{X} \to X$, called the blow up of X at x.

We call $\pi^{-1}(x) = E$ the exceptional divisor. It is biholo to $\mathbb{CP}^{n-1}$, and makes sense as a hypersurface in $\tilde{X}$, so $E \in \mathrm{Div}(\tilde{X})$

Proposition : $[E]|_E = \mathcal{O}(-1)$.

pf: in local coordinates near E, we have $E \cap (\Delta \times U_j) = \{ (z,w) : z = 0 \}$. The transition functions are

$$\psi_{ij}(z,w) = \frac{z_i}{z_j} = \frac{w_i}{w_j},$$

which are exactly the transition functions of $\mathcal{O}(-1)$.

Lemma: $\tilde{X}$ is independent of the coordinate chart φ.

pf: consider $z_j' = f_j(z)$ (z previous coords) new complex coordinates on X. We also need to introduce new w coords. Thinking about it, the role of the w's are lines through the origin in the polydisc, which we can just identify as the tangent space at the origin of the polydisc. Hence w_j' are coordinates of a tangent vector,

$$w_j' = \sum_i \frac{\partial f_j}{\partial z_i}(0)\, w_i$$

Then we have a commutative diagram:

$$\begin{tikzcd}
\tilde{\Delta} \arrow[r, dashed, "F"] \arrow[d, "\sigma"'] & \tilde{\Delta} \arrow[d, "\sigma'"] \\
\Delta \arrow[r, "f"] & \Delta
\end{tikzcd}$$

Claim: $F(z,w) = (z', w')$ as defined above is biholomorphic.

Special case:
Suppose f is linear, given by $[A_j] \in GL(n, \mathbb{C})$. Then $z_i' w_j' = \sum_k A^k_i z_k \sum_\ell A_j{}^\ell w_\ell = \sum_{k,\ell} A^k_i A^\ell_j z_\ell w_k$??

$= z_j' w_i'$

Thus F biholo and diagram commutes

Now wlog assume $\frac{\partial f_i}{\partial z_j}(0) = \delta^i_j$ (complex Jacobian is identity). Then $w_j' = w_j$ $\forall j$. In local coordinates, $\tilde{\Delta}$ (as defined before) have

$$w_1, \dots, \hat{j}, \dots, w_n,\ f_j(z) = z_j + \text{higher order terms}$$

(using f is complex analytic).

Thus $(dF)_p$ = identity of tangent space $\forall p \in \tilde{\Delta}$. By inverse mapping thm (complex variables), F biholo in some nhood of p $\forall p \in \tilde{\Delta}$ $\Rightarrow$ F biholo. ☐

Proposition: let $\sigma: \tilde{X} \to X$ be the blow up of X at a point $x \in X$. Then $K_{\tilde{X}} = \sigma^*(K_X) \otimes [(n-1)E]$, where $n = \dim(X)$.

proof: Assume K_X admits nontrivial mero sections. So let ω be a mero nontrivial $(n,0)$-form on X.

Zeros and poles of pullback $\sigma^*\omega$: away from E, zeros and poles on $\tilde{X}$ are biholomorphically related to those of ω with the same orders.
Near $x \in X$, we have in local coords $\omega = f\, dz_1 \wedge \dots \wedge dz_n$, f holo (mero section), i.e. $\bar{\partial} f = 0$. In local coords on U_j

$$\sigma|_{U_j} : (v_1, \dots, v_{n-1}, z) \longmapsto (z v_1, \dots, \underset{\substack{\uparrow \\ j^{th}\text{ position}}}{z}, \dots, z v_n)$$

(equivalent to first local coord way we wrote it, but not exactly the same)

$$\Rightarrow \quad \sigma^* \omega = (f \circ \sigma)\, d(z v_1) \wedge d(z v_2) \wedge \dots \wedge \underset{\text{jth pos.}}{dz} \wedge \dots \wedge d(z v_{n-1})$$

$$= (f \circ \sigma)\, z^{n-1}\, dv_1 \wedge \dots \wedge dz \wedge \dots \wedge dv_{n-1}$$ by product rule and antisymmetry.

Have an "extra" zero of order $n-1$, along $E \cap U_j = \{ z = 0 \}$. Patching along all j's, this gives us the proposition.

$\square$

Definition: $\forall$ complex manifolds X, define $c_1(X) := -c_1(K_X)$, the "first chern class of X".

Corollary: $c_1(\tilde{X}) = \sigma^* c_1(X) - (n-1)\, \text{P.D.}\, [E]$.

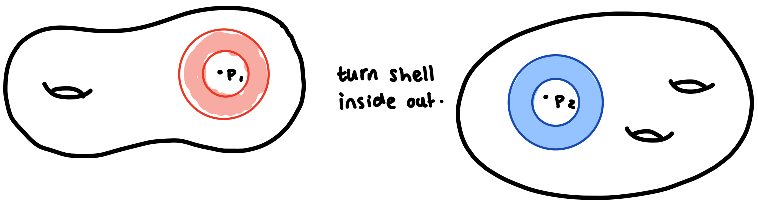
Remark (for topologists) When $\dim_{\mathbb{C}} X = 2$, then $\deg([E]|_E) = -1 = \int_E c_1([E]) = E \cdot E$, the self-intersection number

Blow up as a connected sum

Let M_1 and M_2 be smooth real manifolds with $\dim M_1 = \dim M_2 = m$. Choose a point $p_1 \in M_1$, $p_2 \in M_2$, and take charts $\varphi_i : U_i \subset M_i \to \mathbb{R}^m$ near p_i, wlog $\operatorname{im}(U_i) = B_0 = \{ x \in \mathbb{R}^m, \|x\| < 3 \}$.

Define a map by $\xi : x \in \{ \frac{1}{2} < \|x\| < 2 \} \subset B_0 \longrightarrow \frac{x}{\|x\|^2} \in \{ \frac{1}{2} < \|x\| < 2 \} \subset B_0$, a diffeo of a spherical shell.

Then



The connected sum of M_1 and M_2 at p_1, p_2 is

$$M_1 \# M_2 := \left(M_1 \setminus \varphi_1^{-1}(\|x\| \leq 1/2) \right) \underset{\varphi_2^{-1} \circ \xi \circ \varphi_1}{\bigcup} \left(M_2 \setminus \varphi_2^{-1}(\|x\| < 1/2) \right)$$

Two manifolds connected by a tubular region diffeo to $S^{m-1} \times I$

This is independent of the charts φ_1, φ_2. If we assume that M_1 and M_2 are both oriented and φ_1 preserves the orientation, φ_2 reverses orientation, then since ξ is an orientation reversing diffeo (easy to see), then $M_1 \# M_2$ is oriented with appropriate choices of oriented atlases for M_1 and M_2.

Proposition: let X be a complex manifold of dimension n. Then the blow up $\tilde{X}$ of X at $x \in X$ is diffeo (as a smooth real manifold) $X \# \overline{\mathbb{CP}^n}$ at x and any point in $\overline{\mathbb{CP}^n}$, where $\overline{\mathbb{CP}^n}$ is the underlying real manifold for $\mathbb{CP}^n$, but with orientation reversed from that of the complex structure.

if n odd, conjugate each coord reverses orientation, so your $\overline{\mathbb{CP}^n}$ is actually what you think it is.
But if n even, conjugate even # of times gets you back to same orientation, so not the right idea.
By $\overline{\mathbb{CP}^n}$ we just mean orientation reversed, but if n odd then its conjugate

proof: wlog assume $X = \Delta \subset \mathbb{C}^n$ a polydisc with sufficiently large radius and $0 \in X$ the blow up point. (can just work in a neighbourhood w/ sufficient radius). To show: $\tilde{\Delta}$ is orientation - preserving diffeomorphic to $\overline{\mathbb{CP}}^n \setminus$ (coord ball), the ball that we're going to blow up.

In local coords, $\tilde{\Delta} = \{ (z, w) \in \Delta \times \mathbb{CP}^{n-1} : z_i w_j = z_j w_i \ \forall i, j \}$.

$\overline{\mathbb{CP}}^n = \{ [\bar{z}_0 : z] \mid z_0 \in \mathbb{C}, z \in \mathbb{C}^n, |z_0|^2 + \|z\|^2 \neq 0 \}$

Using a coord chart $\varphi : U = \{ [1 : z] \} \longrightarrow \mathbb{C}^n$; $1 : z \longmapsto z$ is a holo orientation - reversing chart on $\overline{\mathbb{CP}}^n$

Consider $\overline{\mathbb{CP}}^n \setminus \underbrace{\varphi^{-1}(\|z\| < 1/2)}_{K} = \{ [\bar{z}_0 : z] : \|z\| > \frac{1}{2} |z_0| \}$. The gluing map for the connected sum:

$$\Psi : [\bar{z}_0 : z] \in \overline{\mathbb{CP}}^n \setminus K \xrightarrow{\text{diffeo}} \Big(\underbrace{\frac{z_0}{\|z\|^2} z}_{\in \mathbb{C}^n}, \underbrace{\pi(z)}_{\in \mathbb{CP}^{n-1}} \Big) \qquad \pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1} \text{ projection map}$$

$$\in \sigma^{-1}(\|z\| < 2) \subset \tilde{\Delta} .$$

where $\sigma : \tilde{\Delta} \to \Delta$ blow up map

3. Hermitian and kähler Geometry

Definition: a Hermitian metric on a complex manifold X is a (positive - definite) Hermitian inner product h on the (fibres of) the holomorphic tangent bundle, i.e.

$$h(p) : T^{1,0}_p \times T^{1,0}_p \longrightarrow \mathbb{C} \quad \text{with smooth dependence on } p \in X,$$

i.e. $\forall$ smooth sections A, B of $T^{1,0}X$, we have $h(A,B) \in C^\infty(X)$. ($\leftarrow$ complex)

In local coords, $h = \sum_{i,j} h_{i\bar{j}}(z)\, dz_i\, d\bar{z}_j$, with smooth coefficients $h_{i\bar{j}}(z)$.

$\bar{j}$ is a notational convention

If $A = \sum_i A_i \frac{\partial}{\partial z_i}$, $B = \sum_j B_j \frac{\partial}{\partial z_j}$, then $h(A,B) = \sum_{i,j} h_{i\bar{j}} A_i \overline{B_j}$

Proposition: There is a natural equivalence between

- Hermitian metrics on X , and
- J - invariant Riemannian metrics g on the underlying real manifold of X, i.e.

$$g(JA, JB) = g(A,B), \quad \text{where} \quad J \in \Gamma(\operatorname{End} TX^{\mathbb{R}}) \text{ is the almost complex structure.}$$

Proof: Recall $e \in T_x X \xrightarrow{\gamma} e - iJe \in T^{1,0}_x X$ is a linear isomorphism of real vector spaces. Precisely, $\gamma(Je) = i\gamma(e)$. This implies that, given h as above, can construct a Riemannian metric

$$g(u,v) := \tfrac{1}{2} \operatorname{Re}\big(h(u - iJu, v - iJv)\big)$$

Since $h(iA, iB) = h(A,B)$, then $g(Ju, Jv) = g(u,v)$.

For the converse, given g a J-invariant Riemannian metric , we can extend g to a Hermitian h on $TX \otimes_{\mathbb{R}} \mathbb{C}$, given by $h(\lambda u, \mu v) := \lambda \bar{\mu}\, g(u,v)$ $\forall\ u,v \in TX$, $\lambda, \mu \in \mathbb{C}$. Then restrict to the subspaces $T^{1,0}_x X \subset T_x X \otimes \mathbb{C}$, which inverts the first construction.

$\square$

In coordinates, $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$, $g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) \underset{J \text{ invariance}}{=} g\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}\right) = 2 \operatorname{Re} h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right)$

Therefore we can use the concepts of Riemannian Geometry for Hermitian manifolds (a complex manifold equipped with a Hermitian metric) (X,h) via the J-invariant Riemannian metric (the "Re" of h).

Proposition / Definition : Define $\omega(u,v) = -\frac{1}{2} \operatorname{Im} h(u - iJu, v - iJv)^{(*)}$. Then ω is a real $(1,1)$ -form , called the fundamental form of h. Furthermore,

$$\omega(u,v) = g(Ju, v)$$

In fact, any two of ω, g and J determine the remaining one.

proof: $\omega \in \Omega^{1,1} \Leftrightarrow \omega$ is J-invariant : $\omega(Ju, Jv) = \omega(u,v)$. (other form types have different terms from this. In $(*)$, the Ju, Jv in LHS are converted to multiplication by i in RHS. Since h is Hermitian, it's i invariant and so the expression follows.

For the expression $\omega(u,v) = g(Ju,v)$,

Consider $-\frac{1}{2}\operatorname{Im} h(u-iJu, v-iJv) \underset{\substack{\text{first factor}\\ \times i}}{=} \frac{1}{2}\operatorname{Re} h\big(i(u-iJu), v-iJv\big) = \frac{1}{2}\operatorname{Re} h(Ju+iu, v-iJv) = g(Ju,v).$ $\square$

Last part - exercise (easy)

In coords,

$$g = 2\sum_{i,j}\Big((\operatorname{Re} h_{i\bar{j}})(dx_i dx_j + dy_i dy_j) + \operatorname{Im}(h_{i\bar{j}})(dx_i dy_j - dx_i dy_j)\Big)$$

as $g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = 2\operatorname{Re} h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = 2\operatorname{Re}\left(-ih\left(\frac{\partial}{\partial z_i}, i\frac{\partial}{\partial z_j}\right)\right)$

$$= 2\operatorname{Im} h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$$

$$= -\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

$$= -g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_j}\right)$$

Lemma: In complex local coords, $\omega = i\sum_{i,j} h_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$ invariant under complex conjugation, real form.

proof: $i\, dz_i \wedge d\bar{z}_j = i(dx_i + i dy_i) \wedge (dx_j - i dy_j)$

$$= i(dx_i \wedge dx_j + dy_i \wedge dy_j) + (dx_i \wedge dy_j + dx_j \wedge dy_i)$$

Then $\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) = g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = 2\operatorname{Re} h_{i\bar{j}}$.

$\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_j}\right) = -2\operatorname{Im} h_{i\bar{j}} = \omega\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_i}\right)$ $\swarrow$ denote $\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)' := \left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)$

Thus $\sum_{i,j} i h_{i\bar{j}}\, dz_i \wedge d\bar{z}_j = \sum_i h_{i\bar{i}}\, i\, dz_i \wedge d\bar{z}_j + \sum_{i<j} 2\operatorname{Re}\left(h_{i\bar{j}}\, i\, dz_i \wedge d\bar{z}_j\right)$

$$= \sum_{i,j} 2\operatorname{Re} h_{i\bar{j}}\, dx_i \wedge dy_j - \sum_{i<j} 2\operatorname{Im} h_{i\bar{j}}\,(dx_i \wedge dx_j + dy_i \wedge dy_j)$$

$$= i\sum_{i,j} h_{i\bar{j}}\, dz_i \wedge d\bar{z}_j.$$ $\square$

From our work, it follows that for any $a \in T^{1,0}X$, we have $-\omega(a,a) > 0$ for $a \neq 0$. Call any real $(1,1)$-form σ s.t $-i\sigma(a,\bar{a}) > 0 \;\forall\, a \in T^{1,0}\setminus\{0\}$ a positive $(1,1)$-form, which we denote by $\sigma > 0$.

Further, the 1st Chern class of a complex line bundle L say, $c_1(L) > 0$ iff $c_1(L)$ is represented by a (closed) positive $(1,1)$ form.

E.g. if X has $c_1(X) > 0$, then X is called a Fano manifold. If $c_1(X) = 0$ (rep by exact form), then X is called a Calabi-Yau manifold. E.g. $\mathbb{CP}^n$ is a Fano manifold

E.g. complex torus is a Calabi-Yau manifold.

Any positive (1,1) - form is equivalent to a Hermitian metric on X.

deriv. of f is an injective linear map at all points.

If $f: Y \to X$ is a holomorphic immersion (Y is an immersed complex submanifold). Then $f^* g$ is a well-defined Riemannian metric. (I.e. $(df)^{\mathbb{C}} : T^{1,0}_y \to T^{1,0}_{f(y)} X$ injective $\forall\ y \in Y$).

and g is J-invariant, so $df \circ J_X = J_Y \circ df$ converts al. comp of Y to one of X. Therefore, a Hermitian metric is induced on any immersed complex submanifold of X.

If (X, h) is a Hermitian metric, $Y \subset X$ a submanifold, then Y inherits a Hermitian metric by pulling back via the immersion.

Locally, Y is given by the vanishing of n-k coords: $\{z_{k+1} = \cdots = z_n = 0\}$ ($\dim Y = k$ and $\dim X = n$). Then the immersion $f: Y \to X$ is $f(z_1, \dots, z_k) = (z_1, \dots, z_k, 0, \dots, 0)$. Therefore

lemma: the fundamental form of $f^* h$ is $f^* \omega = i \sum_{i,j=1}^{k} h_{ij}\, dz_i \wedge d\bar{z}_j$ (sum only up to k).

Can equivalently give a Hermitian manifold as (X, ω) using the fundamental form (remember two of g, J, ω determine the third).

Definition: a Hermitian manifold (X, ω) with $d\omega = 0$ is called a Kähler manifold. Then ω is called a kähler form on X, and h is a kähler metric.

Examples:

0. $\mathbb{C}^n$, $h = \frac{1}{2} \sum_j dz_j \otimes d\bar{z}_j$ the standard Hermitian metric (real part is Euclidean metric)

$$\omega = \frac{i}{2} \sum_j dz_j \wedge d\bar{z}_j = \sum_j dx_j \wedge dy_j \quad \text{where} \quad z_j = x_j + i y_j$$

Which is the standard symplectic form on $\mathbb{R}^{2n}$.

1. a) the metric in 0. descends to any complex torus $\mathbb{C}^n / \Lambda$, $\Lambda \cong \mathbb{Z}^{2n}$ a discrete lattice in $\mathbb{C}^n$.

b) On a Riemann surface, any non-vanishing 2 - form (compatible with orientation of the complex structure). By dim reasons, it must be a (1,1) form and closed. Its also positive, >0 by compatibility. Hence every Riemann surface is Kähler (with any Hermitian metric)

2. $\pi: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$, $[z_0 : \cdots : z_n]$ coordinates. Consider V_j an affine hyperplane,

$$V_j = \{ z \in \mathbb{C}^{n+1} : z_j = 1 \} \quad \text{an affine hyperplane}$$

Then set $\pi(V_j) \underset{\substack{\uparrow \\ \text{biholo}}}{=:} U_j \subset \mathbb{CP}^n$ as usual.

Then let $\omega = \frac{i}{2\pi} \partial \bar{\partial} \log(\|z\|^2) \in \Omega^{(1,1)}(V_j)$

↑ standard Euclidean norm on V_j

This defines a real (1,1) form on U_j, since $\pi: V_j \to U_j$ is a biholomorphism.

Change of local coordinates $z \in V_j \rightarrow fz \in V_k$, $f = \frac{z_j}{z_k}$ is a holo nonvanishing function on $\pi^{-1}(\pi(V_j) \cap \pi(V_k))$.

On the intersection,

$$\frac{i}{2\pi} \partial\bar{\partial} \log \|fz\|^2 = \frac{i}{2\pi} \partial\bar{\partial} \Big(\log \|z\|^2 + \log \underset{\substack{\uparrow \\ f\bar{f} = \|f\|^2}}{f\bar{f}} \Big) = \omega + \underbrace{\frac{i}{2\pi} \partial\bar{\partial} \log (f\bar{f})}_{\text{claim: this} = 0.}$$

Now consider

$$\frac{i}{2\pi} \partial\bar{\partial} \log (f\bar{f}) = \frac{i}{2\pi} \partial \frac{\bar{\partial}(f\bar{f})}{f\bar{f}} = \omega$$

$$\text{But} \quad \partial \frac{f\bar{\partial}f}{f\bar{f}} = - \frac{\bar{\partial}(\overset{\partial\bar{f} = 0}{\partial\bar{f}})}{\bar{f}} = 0$$

Thus ω is a well-defined form on $\mathbb{CP}^n$.

Also, $\forall\, T \in U(n+1)$, T induces a map $T: \mathbb{CP}^n \rightarrow \mathbb{CP}^n$ called a projective transformation, and $T^*\omega = \omega$.

Lemma / Definition : $\omega > 0$. Thus ω is the fundamental form of a kähler metric on $\mathbb{CP}^n$, called the Fubini - Study metric.

(U(n+1) symmetry)

proof: By our above remark, it suffices to check the positivity at one point.

$$\text{Then} \quad \omega|_{U_0} = \frac{i}{2\pi} \log \|z\|^2 = \frac{i}{2\pi} \partial\bar{\partial} \log \Big(\underset{\substack{\uparrow \\ \text{zero coord is 1}}}{1} + \sum_{j=1}^{n} z_j \bar{z}_j \Big)$$

$$= \frac{i}{2\pi} \partial \sum \frac{z_j \, d\bar{z}_j}{1 + \sum z_j \bar{z}_j}$$

$$= \frac{i}{2\pi} \left(\sum \left(\frac{dz_j \wedge d\bar{z}_j}{1 + \sum z_j \bar{z}_j} \right) - \frac{\sum \bar{z}_j \, dz_j \wedge \sum z_j \, d\bar{z}_j}{(1 + \sum z_j \bar{z}_j)^2} \right)$$

We can just check positivity at e.g. $[1:0:0 \cdots :0]$, i.e. $z_1 = z_2 = \cdots = 0$. Plugging in,

$$= \frac{i}{2\pi} \Big(\sum dz_j \wedge d\bar{z}_j \Big),$$

Which is just example 0, which is positive. Then by symmetry, its positive everywhere. $\square$

3. Given a positive (1,1) - form representing $c_1(X)$ for a Fano manifold X, we can make it into a kähler manifold. In particular, $\mathbb{CP}^n$ is Fano (sheet 4).

4. Proposition : Every complex submanifold of a kähler manifold is kähler (by pulling back kähler form)

Corollary: every projective manifold is kähler.

Detour to Riemannian Geometry

If (M,g) is an oriented Riemannian manifold, $\dim_{\mathbb{R}} M = n$. Then $\forall x \in M$ can use Gram-Schmidt to construct a local orthonormal coframe field $\omega_1, \dots, \omega_n$. Taking the wedge gives the volume form:

Then $$d\mathrm{vol}_g = \omega_1 \wedge \dots \wedge \omega_n$$ positively oriented so dvolg compatible w/ orientation.

is independent of choice of ω_i's and is well-defined, $d\mathrm{vol}_g \in \Omega^k(M)$.

This is called the volume form of (M,g).

Now let (X,h) be a Hermitian manifold, $h = \sum_{i,j} h_{i\bar{j}}\, dz_i \otimes d\bar{z}_j$. Let g be the corresponding Riemannian metric, i.e.

$$g = 2\mathrm{Re}\, h = 2\sum \Big((\mathrm{Re}\, h_{i\bar{j}})(\mathrm{Re}\, dz_i\, d\bar{z}_j) - (\mathrm{Im}\, h_{i\bar{j}})(\mathrm{Im}\, dz_i\, d\bar{z}_j) \Big)$$

$$\omega(\cdot,\cdot) = g(J\cdot,\cdot) \quad \text{the fundamental form.}$$

Near each $x \in X$, we can find "adapted" local orthonormal coframe field $\omega_1, \varepsilon_1, \dots, \omega_n, \varepsilon_n$ for T^*X wrt. g, where $\varepsilon_k = -J\omega_k$, and $\omega_k = J\varepsilon_k$ $\forall k$. Then $\omega_1 + i\varepsilon_k, \dots, \omega_n + i\varepsilon_n$ is an orthonormal coframe field wrt. h for $(T^*X)^{1,0}$.

$$h = \frac{1}{2}\sum_k (\omega_k + i\varepsilon_k) \otimes (\omega_k - i\varepsilon_k), \qquad g = \sum_k (\omega_k \otimes \omega_k + \varepsilon_k \otimes \varepsilon_k)$$

N.B. $J(\omega_k + i\varepsilon_k) = -\varepsilon_k + i\omega_k = i(\omega_k + i\varepsilon_k)$. Thus $\omega_k + i\varepsilon_k$ is a $(1,0)$-form.

Hence $\omega = \sum_k (\omega_k \otimes \varepsilon_k - \varepsilon_k \otimes \omega_k) = \sum_k \omega_k \wedge \varepsilon_k$ (antisymmetric)

In particular, $\omega^n = n!\, \omega_1 \wedge \varepsilon_1 \wedge \omega_2 \wedge \varepsilon_2 \wedge \dots \wedge \omega_n \wedge \varepsilon_n$. Thus we have shown

Proposition: $\boxed{d\mathrm{vol}_g = \frac{\omega^n}{n!}}$ is the volume form of a Hermitian manifold (X,ω).

Consider a complex submanifold $Y \subset X$, $\dim Y = d$. Then $\omega|_Y$ is the fundamental form of the induced Hermitian metric on Y.

Then $\frac{\omega^d}{d!}$ is the volume form of the corresponding Riemannian metric on Y. Obtain following Corollary:

Wirtinger Theorem: for each compact complex submanifold Y of a Hermitian manifold (X,ω).

$$\mathrm{vol}(Y) = \frac{1}{d!}\int_Y \omega^d$$

Suppose that X is compact and Kähler, i.e. $d\omega = 0$, $[\omega] \in H^2_{dR}(X)$. Then

$$\int_X \omega^n = n!\, \mathrm{vol}(X) \neq 0 \qquad (\text{topologically } \langle [\omega]^{\cup n}, [X] \rangle \neq 0)$$

So $[\omega] \neq 0$ by Stoke's theorem (cannot be exact), and for the same reason, $[\omega^k] \neq 0$ $\forall k = 1, \dots, n$ complex. dim. of X.

If Y is a compact complex submanifold, (i.e. $Y^{\mathbb{R}}$ is a closed submanifold), then $[Y] \subset H_{2d}(X, \mathbb{R})$, and

$$\int_Y \omega^d = d!\, vol(Y) \neq 0$$

Therefore $[Y] \neq 0$ in $H_{2d}(X, \mathbb{R})$ by application of Stokes' Theorem. Consequently, taking $X = \mathbb{CP}^n$, we find that for $Y \subset X$ a projective manifold, then $[Y] \neq 0$.

Hodge Theory

Let M be an oriented Riemannian manifold, $\dim_{\mathbb{R}} M = m$, with Riemannian metric g

Then the inner product on $T^*_x M$ defined by the metric can be extended to to the r^{th} exterior power $\Lambda^r T_x^* M$, $x \in M$ so that

$$\left\{ \omega_{i_1} \wedge \dots \omega_{i_r} \;\middle|\; 1 \leq i_1 < \dots < i_r \leq m \right\}$$

is an orthonormal basis (ω_i are a local orthonormal coframe field around x).

In particular, $dvol_g = \omega_1 \wedge \dots \wedge \omega_m$ has norm 1.

Definition: the Hodge $* : \Lambda^p T^*_x M \longrightarrow \Lambda^{m-p} T^*_x M$ is a linear map satisfying

$$\boxed{\alpha \wedge * \beta = \langle \alpha, \beta \rangle_g \; dvol_g} \qquad \forall \alpha, \beta \in \Lambda^p T^*_x M$$

Hodge $*$ is uniquely determined by $*(\omega_{i_1} \wedge \dots \wedge \omega_{i_r}) = \omega_{j_1} \wedge \dots \wedge \omega_{j_{m-r}}$ such that

$$\{ i_1, \dots, i_r, j_1, \dots, j_{m-r} \} = \{ 1, \dots, m \}$$

In particular, $\begin{pmatrix} i_1, \dots, i_r, j_1, \dots, j_{m-r} \\ 1, \quad \dots \quad, m \end{pmatrix}$ is an even permutation

$$\Rightarrow \quad *^2 \alpha = (-1)^{r(m-r)} \alpha$$

note: $dvol_g \mapsto 1$, $1 \mapsto dvol_g$

Remark: can extend $*$ smoothly to $* : \Omega^r(M) \to \Omega^{m-r}(M)$

Now let (X, h) be a Hermitian complex manifold ($\dim_{\mathbb{C}} = n$), and g the corresponding Riemannian metric, invariant under the almost complex structure J. Real dimension $m = 2n$, and so

$$* : \Omega^r(X) \to \Omega^{2n-r}(X)$$

We have as before $\omega_1, \varepsilon_1, \dots, \omega_n, \varepsilon_n$ an adapted local coframe field (adapted $\Rightarrow J\omega_k = -\varepsilon_k$, $J\varepsilon_k = \omega_k \; \forall k$). Then we saw $(1,0)$-forms are spanned by $\langle \omega_k + i\varepsilon_k \rangle_{k=1,\dots,n}$, and $(0,1)$ by $\langle \omega_k - i\varepsilon_k \rangle_{k=1,\dots,n}$.

h is the Hermitian extension of g, so

$$\left|(w_{k_1} + i\varepsilon_{k_1}) \wedge \dots \wedge (w_{k_p} + i\varepsilon_{k_p}) \wedge (w_{\ell_1} - i\varepsilon_{\ell_1}) \wedge \dots \wedge (w_{\ell_q} - i\varepsilon_{\ell_q})\right|^2 \swarrow \text{Hermitian norm}$$

$$= 2^{p+q}$$

each term has norm 2, and there are p+q factors.

We have the induced Hermitian inner product on $\Lambda^{p,q} T^* X$ $\forall p, q$. (still denoted by h) . We can extend $*$ $\mathbb{C}$- linearly ,

$$* : (\Lambda^p T^* X)^{\mathbb{C}} \to (\Lambda^{2n-p} T^* X)^{\mathbb{C}}$$

Lemma: $\forall$ complex differential r- forms α, β, we have

$$\alpha \wedge * \bar{\beta} = \langle \alpha, \beta \rangle_h \, d\,\mathrm{vol}_g$$

pf: Start with α, β real and then multiply by complex constants.

If α, β are real r-forms at $x \in X$, $\lambda, \mu \in \mathbb{C}$, then

$$\langle \lambda\alpha, \mu\beta \rangle_h \, d\mathrm{vol}_g = \lambda\bar{\mu} \langle \alpha, \beta \rangle d\mathrm{vol}_g = \lambda\bar{\mu}\, \alpha \wedge *\beta = \lambda\alpha \wedge * \bar{\mu}\beta$$

Hodge * is real operator,
so commutes with complex mult.

Lemma now follows by linearity of Hermitian prod, wedge and *. □

Corollary: $* : \Omega^{p,q}(X) \to \Omega^{n-q, n-p}(X)$

Corollary: $*^2 \big|_{\Omega^{p,q}(X)} = (-1)^{p+q}$

Definition : $d^* := - * d * : \Omega^r(X) \longrightarrow \Omega^{r-1}(X)$

The Hodge laplacian is $\Delta := dd^* + d^* d : \Omega^r(X) \to \Omega^r(X)$
Both d^* and Δ extend to $\Omega^r(X)^{\mathbb{C}}$

Δ is also known as Hodge Laplacian , Laplace- Beltrami operator

If $X = \mathbb{C}^n$ with Euclidean metric, then for 0- forms, $\Delta = -4 \sum_{j=1}^n \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_j} = - \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right)$

Definition : $\partial^* = - * \bar{\partial} *$, $\bar{\partial}^* = - * \partial *$

Thus $\partial^* : \Omega^{p,q}(X) \to \Omega^{p-1,q}(X)$
$\bar{\partial}^* : \Omega^{p,q}(X) \longrightarrow \Omega^{p, q-1}(X)$.

And $d^* \big|_{\Omega^{p,q}(X)} = \bar{\partial}^* + \partial^*$.

Then $(\partial^*)^2 = 0$ and $(\bar{\partial}^*)^2 = 0$, and $\partial^* \bar{\partial}^* = - \bar{\partial}^* \partial^*$, $(d^*)^2 = 0$.

Definition : the L^2- inner product $\langle \xi, \eta \rangle_{X,h} := \int_X \langle \xi, \eta \rangle_h \, dvol_g$ (assume RHS finite)

$$= \int_X \xi \wedge * \eta$$

makes $\Omega^r(X)$, $\Omega^{p,q}(X)$ into a pre-hilbert space (missing that they smooth functions do not form a complete space in L^2 norm).

observe $\Omega^r(X)^{\mathbb{C}} = \bigoplus_{p+q=r} \Omega^{p,q}(X)$ orthogonal direct sum wrt. norm at a point in X and also in the L^2 norm.

Proposition: ∂^*, $\bar{\partial}^*$ are formal adjoints of ∂, $\bar{\partial}$ wrt. the L^2 inner product,

i.e. $\int_X \langle \partial\alpha, \beta \rangle \, dvol_g = \int_X \langle \alpha, \partial^* \beta \rangle \, dvol_g$

$\int_X \langle \bar{\partial}\alpha, \beta \rangle \, dvol_g = \int \langle \alpha, \bar{\partial}^* \beta \rangle \, dvol_g$.

$\forall$ compactly supported $\alpha \in \Omega^{p-1,q}(X)$, resp. $\Omega^{p,q-1}(X)$, and $\beta \in \Omega^{p,q}(X)$.

Proof: Do second of relations. We use Stokes theorem :

$$\int_X \langle \bar{\partial}\alpha, \beta \rangle \, dvol_g = \int_X \bar{\partial}\alpha \wedge * \bar{\beta} = \int_X \Big(\bar{\partial} (\underbrace{\alpha \wedge * \bar{\beta}}_{\substack{\text{type } n, m-1 \\ n = \dim X}}) - (-1)^{p+q-1} \alpha \wedge \bar{\partial} * \bar{\beta} \Big)$$

So $\bar{\partial}(\alpha \wedge * \bar{\beta}) = d(\alpha \wedge * \bar{\beta})$.

by Stokes, $= (-1)^{p+q} \int_X \alpha \wedge \bar{\partial} * \bar{\beta} = \int_X \alpha \wedge \overline{\partial * \beta}$

$= \int_X \alpha \wedge * (- * \partial * \beta)$ Check the sign from type

$= \int_X \langle \alpha, \bar{\partial}^* \beta \rangle \, dvol_g$

The other identity is similar. ☐

Corollary:

1) $\int_X \langle d\alpha, \beta \rangle \, dvol_g = \int_X \langle \alpha, d^* \beta \rangle \, dvol_g$

2) $\int_X \langle d^c \alpha, \beta \rangle \, dvol_g = \int_X \langle \alpha, (d^c)^* \beta \rangle \, dvol_g$ - exercise noting $(d^c)^* = -i(\bar{\partial}^* - \partial^*) = - * d^c *$

Definition : $\Delta_\partial = \partial \partial^* + \partial^* \partial$, $\Delta_{\bar{\partial}} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$

Then $\Delta_\partial, \Delta_{\bar{\partial}} : \Omega^{p,q}(X) \rightarrow \Omega^{p,q}(X)$

and by the above, Δ, Δ_∂ and $\Delta_{\bar{\partial}}$ are formally self adjoint.

N.B. Δ in general does <u>not</u> act on (p,q) - forms

Definition: an r-form α is (d-) harmonic, denoted $\alpha \in \mathcal{H}^r(X)$, if $\Delta\alpha = 0$

A (p,q)-form is $\bar\partial$-harmonic $\alpha \in \mathcal{H}^{p,q}_{\bar\partial}(X)$ if $\Delta_{\bar\partial}\alpha = 0$

A (p,q)-form is ∂-harmonic $\alpha \in \mathcal{H}^{p,q}_{\partial}(X)$ if $\Delta_{\partial}\alpha = 0$.

Proposition: Suppose a Hermitian n-manifold X is (necessarily) compact. Then

$$\Delta_{\bar\partial}\alpha = 0 \quad \text{iff} \quad \bar\partial\alpha = 0 \quad \text{and} \quad \bar\partial^*\alpha = 0$$

$$\Delta_{\partial}\alpha = 0 \quad \text{iff} \quad \partial\alpha = 0 \quad \text{and} \quad \partial^*\alpha = 0$$

$$\Delta\alpha = 0 \quad \text{iff} \quad d\alpha = 0 \quad \text{and} \quad d^*\alpha = 0$$

Proposition: $0 = \int_X \langle \Delta_{\bar\partial}\alpha, \alpha\rangle \, dvol_g \underset{\substack{\uparrow \\ \text{propn}}}{=} \int_X \left(\langle \bar\partial\alpha, \bar\partial\alpha\rangle + \langle \bar\partial^*\alpha, \bar\partial^*\alpha\rangle\right)$

Basic analysis says $\Leftrightarrow \quad \langle \bar\partial\alpha, \bar\partial\alpha\rangle = \langle \bar\partial^*\alpha, \bar\partial^*\alpha\rangle = 0$ ☐

Δ_∂ and Δ are similar.

Hodge Theorem

Let (X,h) be a compact Hermitian manifold. Then $\forall r$, $\mathcal{H}^r(X)$, $\forall p,q$ $\overset{\text{equiv } \partial \ \swarrow}{\mathcal{H}^{p,q}_{\bar\partial}}(X)$ are finite dimensional, and then L^2-orthogonal direct sum decompositions

$$\Omega^r(X) = \mathcal{H}^r(X) \oplus d\Omega^{r-1}(X) \oplus d^*\Omega^{r+1}(X)$$

$$\Omega^{p,q}(X) = \mathcal{H}^{p,q}_{\bar\partial}(X) \oplus \bar\partial\Omega^{p,q-1}(X) \oplus \bar\partial^*\Omega^{p,q+1}(X)$$

assume this without proof (requires analysis of PDEs)

$\Delta_{\bar\partial}\alpha = \overline{\Delta_\partial\bar\alpha}$ (hence suffices to look at just $\Delta_{\bar\partial}$).

Applications:

Proposition 1: Assume X is compact and Hermitian.

(1) $\alpha \in \mathcal{H}^r(X) \to [\alpha] \in H^r_{dR}(X)$ is an $\mathbb{R}$-linear isomorphism of vector spaces

(2) if $\alpha \in \mathcal{H}^{p,q}_{\bar\partial}(X) \longrightarrow [\alpha] \in H^{p,q}(X)$ is a $\mathbb{C}$-linear isomorphism of vector spaces

Exercise: or see online notes on PT III diff geo.

Define $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X) = \dim_{\mathbb{C}} \mathcal{H}^{p,q}(X)$, the Hodge numbers of X

In particular, if $q=0$, then $h^{p,0}(X) = \dim(\text{holo. } p\text{-forms on } X)$

$n = \dim_{\mathbb{C}} X$, then $h^{n,0}(X) = \dim(\text{holo sections of canonical bundle } K_X) = P_{\underset{\substack{\uparrow \\ \text{g for geometric, not} \\ \text{from metric}}}{g}}(X)$ is the geometric genus of X

2] $\alpha \in \mathcal{H}^{p,q}_{\bar\partial}(X) \Leftrightarrow *\alpha \in \mathcal{H}^{n-q,n-p}_{\bar\partial}(X) \Longleftrightarrow \overline{*\alpha} = *\bar\alpha \in \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)$ is a linear isomorphism